INTERPOLATION THEORY AND LOCAL MORREY-TYPE SPACES

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Real interpolation method

For $\lambda \in \mathbb{R}$ and $0 < q \leqslant \infty$ denote by $\Phi_{\lambda,q}$ the space of all functions φ, Lebesgue measurable on $(0,\infty)$, for which

$$||\varphi||_{\Phi_{\lambda,p}} = \left(\int\limits_0^\infty \left(t^{-\lambda} |\varphi(t)| \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty,$$

if $q < \infty$, and

$$||\varphi||_{\Phi_{\lambda,\infty}} = \text{ess sup }_{x \in (0,\infty)} t^{-\lambda} |\varphi(t)| < \infty,$$

if $q = \infty$.

Moreover, let Z_0, Z_1 be quasi-normed linear subspaces of a linear space Z. For $0 < q \leqslant \infty,\ 0 < \theta < 1$ by $(Z_0, Z_1)_{\theta,q}$ we denote the interpolation space of all functions $f \in Z_0 + Z_1$, for which

$$||f||_{(Z_0,Z_1)_{\theta,q}} = ||K(Z_0, Z_1, f)||_{\Phi_{\theta,q}} < \infty,$$

where $K(Z_0, Z_1, f)$ is the so-called K-functional, defined for $t > 0$

$$K(Z_0, Z_1, f)(t) = \inf_{\substack{f = f_0 + f_1 \\ f_0 \in Z_0,\ f_1 \in Z_1}} \left(||f_0||_{Z_0} + t||f_1||_{Z_1} \right).$$

If X_0, X_1 are quasinormed linear subspaces of a linear space X, Y_0, Y_1 are quasi-normed linear subspaces of a linear space Y and T is a linear operator for which

$$T : X_0 \to Y_0, \quad T : X_1 \to Y_1,$$

and T is bounded in both cases, then, for any $0 < q \leqslant \infty,\ 0 < \theta < 1$,

$$T : (X_0, X_1)_{\theta,q} \to (Y_0, Y_1)_{\theta,q}$$

and T is bounded.

This fact explains the importance of describing the interpolation spaces for a given pair of quasi-normed spaces.

Interpolation of Morrey spaces

Let $0 < p \leqslant \infty$ and $0 \leqslant \lambda \leqslant \frac{n}{p}$. The Morrey spaces M_p^λ are the spaces of all functions $f \in L_p^{loc}(\mathbb{R}^n)$ such that

$$\|f\|_{M_p^\lambda} = \sup_{x\in\mathbb{R}^n} \sup_{r>0} r^{-\lambda} \|f\|_{L_p(B(x,r))} < \infty \,,$$

where $B(x,r)$ is the open ball of radius $r > 0$ with center at point $x \in \mathbb{R}^n$.

In other words a function f belongs to the Morrey space M_p^λ if $f \in L_p^{loc}(\mathbb{R}^n)$ and there exists $c = c(f) > 0$ such that

$$\|f\|_{L_p(B(x,r))} \leqslant c\, r^\lambda$$

for all $r > 0$ and for all $x \in \mathbb{R}^n$. (The minimal value of c is $\|f\|_{M_p^\lambda}$).

If $\lambda = 0$, then $M_p^0 = L_p(\mathbb{R}^n)$, while if $\lambda = \frac{n}{p}$, then $M_p^{\frac{n}{p}} = L_\infty(\mathbb{R}^n)$. If $\lambda < 0$ or $\lambda > \frac{n}{p}$, then $M_p^\lambda = \Theta$, where Θ is the set of all functions that are equivalent to zero on $\mathbb{R}^n$.

It follows by the results of J. Peetre (1969) that

$$(M_p^{\lambda_0}, M_p^{\lambda_1})_{\theta,\infty} \subset M_p^\lambda \,,$$

where $\lambda = (1-\theta)\lambda_0 + \theta\lambda_1$.

O. Blasco, A. Ruiz, L. Vega (1999) established that this inclusion is strict, hence

the scale of Morrey spaces M_p^λ

is not closed under procedure of interpolation

.

Interpolation of local Morrey spaces

Consider a local variant of the Morrey spaces. Let $0 < p \leqslant \infty$ and $\lambda \geqslant 0$. The local Morrey spaces LM_p^λ are the spaces of all functions $f \in L_p^{loc}(\mathbb{R}^n)$ such that

$$\|f\|_{LM_p^\lambda} = \sup_{r>0} r^{-\lambda} \|f\|_{L_p(B(0,r))} < \infty \,,$$

In other words a function f belongs to the Morrey space M_p^λ if $f \in L_p^{loc}(\mathbb{R}^n)$ and there exists $c = c(f) > 0$ such that

$$\|f\|_{L_p(B(0,r))} \leqslant cr^\lambda$$

for all $r > 0$. (The minimal value of c is $\|f\|_{LM_p^\lambda}$).

Theorem 1. *Let* $0 < p \leqslant \infty$ *and* $0 < \theta < 1$. *Then*

$$(LM_p^{\lambda_0}, \;\; LM_p^{\lambda_1})_{\theta,\infty} = LM_p^\lambda,$$

where $\lambda = (1-\theta)\lambda_0 + \theta\lambda_1$.

So, in contrast to the scale of the Morrey spaces M_p^λ

the scale of local Morrey spaces LM_p^λ

is closed under the procedure of interpolation

.

Interpolation of local Morrey-type spaces

Consider a more general local variant of the Morrey spaces. The local Morrey-type spaces $LM^{\lambda}_{p,q}$ are defined for $\lambda \geqslant 0$, and $0 < p, q \leqslant \infty$ as the spaces of all functions $f \in L^{loc}_p(\mathbb{R}^n)$ such that

$$\|f\|_{LM^{\lambda}_{p,q}} = \Big\| \|f\|_{L_p(B(0,r))} \Big\|_{\Phi_{\lambda,q}} = \left(\int_0^{\infty} \left(r^{-\lambda} \|f\|_{L_p(B(0,r))} \right)^q \frac{dr}{r} \right)^{\frac{1}{q}} < \infty \, ,$$

with the conventional modification for $q = \infty$.

Note that $LM^{\lambda}_{p,q} \neq \Theta$ if and only if $\lambda > 0$ for $q < \infty$ and $\lambda \geqslant 0$ for $q = \infty$. If $q = \infty$, then $LM^{0}_{p,\infty} = L_p(\mathbb{R}^n)$. Moreover, for $p = q$, we have

$$LM^{\lambda}_{p,p} = L_{p,\lambda}(\mathbb{R}^n)$$

and

$$\|f\|_{LM^{\lambda}_{p,p}} = (\lambda p)^{-\frac{1}{p}} \|f\|_{L_{p,\lambda}(\mathbb{R}^n)},$$

where $L_{p,\lambda}(\mathbb{R}^n)$ is the weighted Lebesgue space of all functions f Lebesgue measurable on $\mathbb{R}^n$ for which

$$\|f\|_{L_{p,\lambda}(\mathbb{R}^n)} = \|f(y)|y|^{-\lambda}\|_{L_p(\mathbb{R}^n)} < \infty.$$

Theorem 2. *Let $0 < p, q_0, q_1, q \leqslant \infty$ and $0 < \theta < 1$. Suppose, in addition, that $\lambda_0 \neq \lambda_1$ and $\lambda_0, \lambda_1 > 0$ if at least one of the parameters q_0, q_1 and q is finite, and $\lambda_0, \lambda_1 \geqslant 0$ if $q_0 = q_1 = q = \infty$. Then*

$$(LM^{\lambda_0}_{p,q_0}, \;\; LM^{\lambda_1}_{p,q_1})_{\theta,q} = LM^{\lambda}_{p,q},$$

where $\lambda = (1-\theta)\lambda_0 + \theta\lambda_1$.

So,

the scale of local Morrey-type spaces $LM^{\lambda}_{p,q}$

is also closed under the procedure of interpolation

.

Two variants of the definition of general local Morrey-type spaces

One of the most popular definitions of general local Morrey-type spaces is the quite natural following one (the power function $r^{-\lambda}$ is replaced by a general function $w(r)$).

Definition 1. Let $0 < p, q \leqslant \infty$ and let w be a non-negative measurable function on $(0, \infty)$. We denote by $LM_{p,q,w(\cdot)}$ the local Morrey-type spaces the spaces of all functions $f \in L_p^{loc}(\mathbb{R}^n)$ with finite quasinorms

$$\|f\|_{LM_{p,q,w(\cdot)}} = \left\| w(r) \|f\|_{L_p(B(0,r))} \right\|_{L_q(0,\infty)} .$$

However, this definition appeared not to be quite useful for interpolation purposes. More preferable is the following one.

Let (Ω, μ) be a space with a positive σ-finite Borel measure μ and $G = \{G_t\}_{t>0}$ a family of μ-measurable subsets of Ω, for which

$$G_t \neq \Omega \text{ for some } t > 0, \quad G_{t_1} \subset G_{t_2} \text{ if } 0 < t_1 < t_2 < \infty \qquad (1)$$

and

$$\bigcup_{t>0} G_t = \Omega. \qquad (2)$$

Definition 2. Let $0 < p, q \leqslant \infty$ and $0 < \lambda < \infty$ if $q < \infty$ and $0 \leqslant \lambda < \infty$ if $q = \infty$. We define the space $LM_{p,q}^{\lambda}(G, \mu)$ as the space of all functions f μ-measurable on Ω such that for

$$\|f\|_{LM_{p,q}^{\lambda}(G,\mu)} = \left\| \|f\|_{L_p(G_t,\mu)} \right\|_{\Phi_{\lambda,q}} = \left(\int_0^\infty \left(t^{-\lambda} \|f\|_{L_p(G_t,\mu)} \right)^q \frac{dt}{t} \right)^{1/q} < \infty,$$

where

$$\|f\|_{L_p(G_t,\mu)} = \left(\int_{G_t} |f(x)|^p d\mu \right)^{\frac{1}{p}},$$

with the conventional modifications for $q = \infty$ and $p = \infty$.

Relationship between two definitions

Let $0 < p, q \leqslant \infty$, and let w be a positive measurable function on $(0, \infty)$ such that $||w||_{L_q(t,\infty)} < \infty$ for all $t > 0$. Assume that μ is the Lebesgue measure,

$$v(t) = ||w||^{-1}_{L_q(t,\infty)}, \quad t > 0,$$

and

$$G = \{G_t\}_{t>0}\,, \quad \text{where} \quad G_t = B(0, v^{(-1)}(t)), \quad t > 0,$$

Then

$$LM^1_{p,q}(G, \mu) = LM_{pq,w(\cdot)}$$

and

$$||f||_{LM^1_{p,q}(G,\mu)} = ||f||_{LM_{pq,w(\cdot)}} \equiv ||w(r)||f||_{L_p(B(0,r))}||_{L_q(0,\infty)},$$

Moreover, for any $0 < \lambda < \infty$

$$LM^\lambda_{p,q}(G, \mu) = LM_{pq,w^\lambda(\cdot)}$$

and

$$||f||_{LM^\lambda_{p,q}(G,\mu)} = \lambda^{-\frac{1}{q}}||f||_{LM_{pq,w^\lambda(\cdot)}}.$$

Also, if for $0 < \lambda < \infty$

$$v_\lambda(t) = \left\|w^{\frac{1}{\lambda}}\right\|^{-1}_{L_q(t,\infty)}, \quad t > 0,$$

and

$$G_\lambda = \{G_t(\lambda)\}_{t>0}\,, \quad \text{where} \quad G_t(\lambda) = B(0, v_\lambda^{(-1)}(t)), \quad t > 0,$$

then

$$LM_{pq,w(\cdot)} = LM^\lambda_{p,q}(G_\lambda, \mu)$$

and

$$||f||_{LM_{pq,w(\cdot)}} = \lambda^{\frac{1}{q}}||f||_{LM^\lambda_{p,q}(G_\lambda,\mu)}.$$

Interpolation of general local Morrey-type spaces

Theorem 3. Let $0 < p, q_0, q_1, q \leqslant \infty$, $0 < \lambda_0, \lambda_1 < \infty$, $\lambda_0 \neq \lambda_1$, $0 < \theta < 1$. Moreover, let $\Omega \subset \mathbb{R}^n$, μ be a σ-finite Borel measure on Ω and $G = \{G_t\}_{t>0}$ be a family of μ-measurable sets G_t, satisfying conditions (1) and (2).

Then

$$(LM^{\lambda_0}_{p,q_0}(G,\mu), LM^{\lambda_1}_{p,q_1}(G,\mu))_{\theta,q} = LM^{\lambda}_{p,q}(G,\mu),$$

where $\lambda = (1-\theta)\lambda_0 + \theta\lambda_1$.

Moreover, there exist $c_1, c_2 > 0$, depending only on $p, q_0, q_1, q, \lambda_0, \lambda_1$ and θ, such that

$$c_1 ||f||_{LM^{\lambda}_{p,q}(G,\mu)} \leqslant ||f||_{(LM^{\lambda_0}_{p,q_0}(G,\mu), LM^{\lambda_1}_{p,q_1}(G,\mu))_{\theta,q}} \leqslant c_2 ||f||_{LM^{\lambda}_{p,q}(G,\mu)}$$

for all $f \in LM^{\lambda}_{p,q}(G,\mu)$.

So,

the scale of general local Morrey-type spaces $LM^{\lambda}_{p,q}(G,\mu)$ is also closed under the procedure of interpolation

.

Corollary. Let $0 < p, q_0, q_1, q < \infty$, $q_0 \neq q_1$, $0 < \theta < 1$,

$$\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$$

and $w \in \Omega_1$.

Then

$$\left(LM_{p,q_0,w^{\frac{1}{q_0}}(\cdot)}, LM_{p,q_1,w^{\frac{1}{q_1}}(\cdot)}\right)_{\theta,q} = LM_{p,q,w^{\frac{1}{q}}(\cdot)}.$$

Further developments

The analysis of the proof of Theorem 3 showed that a much more general interpolation theorem can be proved by considering the spaces obtained from Definition 2 by replacing $\|f\|_{L_p(G_t,\mu)}$ by an arbitrary functional F such that Ff belongs to the space $M^{\uparrow}$ of all non-negative non-decreasing functions on $(0,\infty)$.

Definition 3. Let X be a linear space, $0 < q \leqslant \infty$, $Z \subset X$ and $F : Z \to M^{\uparrow}$. We say that $f \in \Phi_{\lambda,q}(F)$ if

$$||f||_{\Phi_{\lambda,q}(F)} = ||Ff||_{\Phi_{\lambda,p}} = \left(\int_0^\infty \left(t^{-\lambda} Ff(t)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} < \infty .$$

Let $0 < p \leqslant \infty$. If

$$Ff(t) = \|f\|_{L_p(B(0,t))}$$

then

$$||f||_{\Phi_{\lambda,q}(F)} = LM^{\lambda}_{p,q} .$$

If

$$Ff(t) = \|f\|_{L_p(G_t,\mu)}$$

then

$$\Phi_{\lambda,q}(F) = LM^{\lambda}_{p,q}(G,\mu) .$$

Importantly, there are many other useful choices of F. For example, if

$$Ff(t) = f^*\left(\frac{1}{t}\right), \quad t > 0,$$

where f^* is the non-increasing rearrangement of f, then

$$\Phi_{\frac{1}{p},q}(F) = LM_{p,q} .$$

where $LM_{p,q}$ is the Lorentz space.

General interpolation theorem

We say that F is a *weakly quasi-additive* functional if there exists $c, \alpha \geqslant 1$ such that for every $f_0, f_1 \in Z$

$$F(f_0 + f_1)(t) \leqslant c\left(Ff_0(\alpha t) + Ff_1(\alpha t)\right),\ t > 0.$$

Consider the families $A = \{A_s\}_{s>0}$ $B = \{B_s\}_{s>0}$ of the operators A_s, B_s, defined for $s > 0$ and functions $\varphi \in M^{\uparrow}$ by, $A_s\varphi$ $B_s\varphi$

$$A_s\varphi(t) = \begin{cases} \varphi(t), & 0 < t \leqslant s, \\ \varphi(s), & t > s \end{cases}$$

and

$$B_s\varphi(t) = \varphi(t)\chi_{(s,\infty)}(t),\ t > 0,$$

where $\chi_{(s,\infty)}$ is the characteristic function of the interval (s, ∞).

We say that F admits a *weakly A-B majorisable decomposition* if there exist $c, \alpha \geqslant 1$ and for every $f \in Z$ and $s > 0$ there exist $f_{0,s}, f_{1,s} \in Z$ such that $f = f_{0,s}, +f_{1,s}$ and

$$Ff_{0,s}(t) \leqslant cA_sFf(\alpha t),\ \ Ff_{1,s}(t) \leqslant cB_sFf(\alpha t),\ t > 0.$$

Theorem 4. Let $0 < q_0, q_1, q \leqslant \infty$, $\lambda_0 \neq \lambda_1$, $0 < \theta < 1$, $\lambda = (1-\theta)\lambda_0 + \theta\lambda_1$ and $F : Z \to M^{\uparrow}$.

1. If F is a weakly quasi-additive operator, then

$$\Phi_{\lambda,q}(F) \subset \left(\Phi_{\lambda_0,q_0}(F), \Phi_{\lambda_1,q_1}(F)\right)_{\theta,q}.$$

2. If F admits a weakly A-B majorisable decomposition, then

$$\left(\Phi_{\lambda_0,q_0}(F), \Phi_{\lambda_1,q_1}(F)\right)_{\theta,q} \subset \Phi_{\lambda,q}(F).$$

3. If the assumptions of both Parts 1 and 2 are satisfied, then

$$\left(\Phi_{\lambda_0,q_0}(F), \Phi_{\lambda_1,q_1}(F)\right)_{\theta,q} = \Phi_{\lambda,q}(F).$$

In order to apply Theorem 4 one needs to prove, for a chosen functional F, the properties required in Parts 1 and 2 of that theorem. The property in Part 1 is rather easy to be verified. The main work to be done is proving the property of Part 2.

Theorems 1-3 are particular cases of Theorem 4.

In its turn Theorem 4 is a particular case of still more general interpolation theorem [3], which we do not formulate here (in most applications it suffices to use Theorem 4).

Importantly all classical interpolation theorems due to Stein-Weiss, Peetre, Calderon, Gilbert, Lizorkin, Freitag and some of their new variants can be derived from that theorem.

References

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