Complex interpolation theorem on B^u_w spaces

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The joint work with Denny I. HAKIM
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Central Morrey spaces and its generalizations

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B_σ, $\dot{B}_\sigma$ (Y. Komori-Furuya, K. Matsuoka, E. Nakai and Y. Sawano, 2013)

$$\|f\|_{B_\sigma(E)} = \sup_{r \geq \delta} \frac{1}{r^\sigma} \|f\|_{E(Q_r)} = \left\| \frac{1}{r^\sigma} \|f\|_{E(Q_r)} \right\|_{L^\infty[\delta,\infty)}$$

Definition of $B^u_w(E)$ spaces

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$$\|f\|_{B^u_w(E)} = \left\| w(r)\|f\|_{E(Q_r)} \right\|_{L^u([\delta,\infty), \frac{dr}{r})} = \begin{cases} \left[\int_\delta^\infty \left(w(r)\|f\|_{E(Q_r)} \right)^u \frac{dr}{r} \right]^{1/u} \\ \sup_{r\geq\delta} \left(w(r)\|f\|_{E(Q_r)} \right), \ (u=\infty), \end{cases}$$

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Morrey space, Weak-Morrey space, Campanato space, Lipschitz space

Condition of w

almost decreasing

$$w(r) \geq Cw(s) \quad \text{for} \quad r \leq s.$$

doubling condition

$$C^{-1} \leq \frac{w(r)}{w(s)} \leq C \quad \text{for} \quad \frac{1}{2} \leq \frac{r}{s} \leq 2.$$

$$\mathcal{W}^u = \left\{ w : \text{almost decreasing, doubling, } \int_1^\infty w(r)^u \frac{dr}{r} < \infty \right\}$$

$$\mathcal{W}^* = \left\{ w : \text{almost decreasing, doubling, } \int_r^\infty w(t) \frac{dt}{t} \approx w(r) \right\}$$

$$u_1 \leq u_2 \implies \mathcal{W}^* \subset \mathcal{W}^{u_1} \subset \mathcal{W}^{u_2} \subset \mathcal{W}^\infty.$$

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- *For* $\epsilon > 0$, *either* $\dfrac{w_0(r)}{w_1(r)} r^{-\epsilon}$ *or* $\dfrac{w_1(r)}{w_0(r)} r^{-\epsilon}$ *is almost increasing.*

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$$(\dot{B}^{u_0}_{w_0}(E)(\mathbb{R}^n), \dot{B}^{u_1}_{w_1}(E)(\mathbb{R}^n))_{\theta,u,(0,\infty)} = \dot{B}^u_w(E)(\mathbb{R}^n)$$

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some more assumption of the space E to use K-real interpolation method.

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More, we hope

$$w = w_0^{1-\theta} w_1^\theta, \qquad \frac{1}{u} = \frac{1-\theta}{u_0} = \frac{\theta}{u_1}.$$

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◎ $[X_0, X_1]_\theta \subset [X_0, X_1]^\theta$.

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$$[X_0, X_1]^\theta = \{x \in X_0 + X_1;\ x = G'(\theta), G \in \mathcal{G}(X_0, X_1)\}$$

$$\|x\|_{[X_0,X_1]^\theta}$$

$$= \inf_{G\in\mathcal{G}, x=G'(\theta)} \left(\max\left\{ \sup_{t\in\mathbb{R}} \|G(it)\|_{\mathrm{Lip}(\mathbb{R},X_0)},\ \sup_{t\in\mathbb{R}} \|G(1+it)\|_{\mathrm{Lip}(\mathbb{R},X_1)} \right\} \right)$$

◎ $[X_0, X_1]_\theta \subset [X_0, X_1]^\theta$. If X_0 or X_1 is reflexive, $[X_0, X_1]_\theta = [X_0, X_1]^\theta$.

We treat non-reflexive spaces!

Interpolation of $B_w^u(L^p)$ spaces ($u \neq \infty$)

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Let $u_0, u_1 < \infty$, $1 \leq p_0, p_1 < \infty$ and $p_0 \neq p_1$.

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$$\frac{u_0}{p_0} = \frac{u_1}{p_1}\left(= \frac{u}{p}\right), \quad w_0^{u_0} = w_1^{u_1}(= w^u).$$

Theorem 2

$$[\dot{B}^{u_0}_{w_0}(L^{p_0}), \dot{B}^{u_1}_{w_1}(L^{p_1})]_\theta = [\dot{B}^{u_0}_{w_0}(L^{p_0}), \dot{B}^{u_1}_{w_1}(L^{p_1})]^\theta = \dot{B}^u_w(L^p)$$

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The Case $u_0 = u_1 = u = \infty$ (Local(Central) Morrey spaces)

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Theorem 3

$$[\dot{B}^{\infty}_{w_0}(L^{p_0}), \dot{B}^{\infty}_{w_1}(L^{p_1})]_\theta = \left\{ f \in \dot{B}^{\infty}_{w}(L^p) : \lim_{j \to \infty} \|f - \chi_{\{\frac{1}{j} \le |f| \le j\}} f\|_{\dot{B}^{\infty}_{u}(L^p)} = 0 \right\}$$

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Remark: $\mathcal{M}_q^p \subset B_w^{\infty}(L^p)$, $\mathcal{M}_q^p \not\subset L^1 + L^{\infty}$.

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Calderón product

$\overline{X} = (X_0, X_1)$: compatible couple of Banach spaces of m'ble func. on $\mathbb{R}^n$

Definition 5

$$X_0^{1-\theta} X_1^{\theta} := \bigcup_{f_0 \in X_0, f_1 \in X_1} \{ f : \mathbb{R}^n \to \mathbb{C} \, : \, |f(x)| \le |f_0(x)|^{1-\theta} |f_1(x)|^{\theta} \}.$$

$$\|f\|_{X_0^{1-\theta} X_1^{\theta}} = \inf\{ \|f_0\|_{X_0}^{1-\theta} \|f_1\|_{X_1}^{\theta} \, : \, f_j \in X_j, |f(x)| \le |f_0(x)|^{1-\theta} |f_1(x)|^{\theta} \}$$

(quasi-norm).

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(quasi-norm).

Following relations are known.

$$[X_0, X_1]_\theta = \overline{X_0 \cap X_1}^{[X_0, X_1]^\theta} = \overline{X_0 \cap X_1}^{X_0^{1-\theta} X_1^{\theta}}$$

Calderón product of $\dot{B}^u_w(E)$, $B^u_w(E)$

Lemma 6

$u_0, u_1 < \infty$, $\dfrac{u_0}{p_0} = \dfrac{u_1}{p_1}$ *and* $w_0^{u_0} = w_1^{u_1}$,

$$\left(\dot{B}^{u_0}_{w_0}(L^{p_0})\right)^{1-\theta} \left(\dot{B}^{u_1}_{w_1}(L^{p_1})\right)^{\theta} = \dot{B}^u_w(L^p)$$

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With these lemma, we proved the interpolation theorem above.

$\dot{B}^u_w(\mathcal{M}^p_q)$ and $B^u_w(\mathcal{M}^p_q)$

$1 \le q \le p < \infty$,

$$\|f\|_{\mathcal{M}^p_q} = \sup_{a \in \mathbb{R}^n, r>0} |B(a,r)|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{B(a,r)} |f(x)|^q \, dx \right)^{1/q} < \infty.$$

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Definition 8

$$\|f\|_{B_w^u(\mathcal{M}_q^p)} := \begin{cases} \left(\int_\delta^\infty (w(r)\|f\|_{M_q^p(B(r))})^u \frac{dr}{r} \right)^{\frac{1}{u}}, & \text{for } u < \infty \\ \sup_{r \ge \delta} w(r)\|f\|_{M_q^p(B(r))}, & \text{for } u = \infty. \end{cases}$$

When $\delta = 0+$, we rewrite it $\dot{B}_w^u(\mathcal{M}_q^p)$.

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Lemma 9

$$\textit{When } \frac{p_0}{q_0} = \frac{p_1}{q_1}, \quad \dot{B}_{w_0}^{u_0}(\mathcal{M}_{q_0}^{p_0})^{1-\theta} \dot{B}_{w_1}^{u_1}(\mathcal{M}_{q_1}^{p_1})^{\theta} = \dot{B}_w^u(\mathcal{M}_q^p)$$

Interpolation of $\dot{B}^u_w(\mathcal{M}^q_p)$ and $B^u_w(\mathcal{M}^q_p)$

Theorem 10

Assume that

$$\frac{p_0}{q_0} = \frac{p_1}{q_1}, \quad \frac{u_0}{q_0} = \frac{u_1}{q_1}, \quad w_0^{u_0} = w_1^{u_1}.$$

Define $u, p, q,$ *and* w *by*

$$\frac{1}{u} := \frac{1-\theta}{u_0} + \frac{\theta}{u_1}, \quad \frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1},$$

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Then we have

$$[\dot{B}^{u_0}_{w_0}(\mathcal{M}^{p_0}_{q_0}), \dot{B}^{u_1}_{w_1}(\mathcal{M}^{p_1}_{q_1})]_\theta = \left\{ f \in \dot{B}^u_w(\mathcal{M}^p_q) : \lim_{j\to\infty} \|f(1-\chi_{\{\frac{1}{j}\le|f|\le j\}})\|_{\dot{B}^u_w(\mathcal{M}^p_q)} = 0 \right\}.$$

The roles of closed subspace

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U: linear subspace of L^0,

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Denote $$U\dot{B}_w^u(L^p) = \overline{U \cap \dot{B}_w^u(L^p)}^{\dot{B}_w^u(L^p)}.$$

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On B^u_w case,

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$$U\dot{B}^u_w(L^p) = \overline{U \cap \dot{B}^u_w(L^p)}^{\dot{B}^u_w(L^p)}.$$

Examples: L^p, Morrey class, compact supported bounded functions, etc.

The roles of closed subspace

Theorem 4

$$[U\dot{B}^{u_0}_{w_0}(L^{p_0}), U\dot{B}^{u_1}_{w_1}(L^{p_1})]_\theta = U\dot{B}^u_w(L^p) \cap [\dot{B}^{u_0}_{w_0}(L^{p_0}), \dot{B}^{u_1}_{w_1}(L^{p_1})]_\theta$$

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Theorem 5

$$[U\dot{B}^{\infty}_{w_0}(L^{p_0}), U\dot{B}^{\infty}_{w_1}(L^{p_1})]_\theta = \left\{ f \in U\dot{B}^{\infty}_{w}(L^{p}) \,:\, \lim_{j\to\infty} \|f - \chi_{\{\frac{1}{j} \le |f| \le j\}} f\|_{\dot{B}^{\infty}_{w}(L^{p})} = 0 \right\}$$

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Thank you for the attention.