# *Generalized Morrey Type Spaces over Unbounded Domains*

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Charls Bradfield Morrey Jr.

23 July 1907 - 29 April 1987

## Definition (Morrey space)

Let $p \in [1, +\infty)$, $\lambda \in (0, n)$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain. The space $L^{p,\lambda}(\Omega)$ is the set of all functions $f \in L^p(\Omega)$ such that

$$\|f\|^p_{L^{p,\lambda}(\Omega)} = \sup_{\substack{x\in\Omega \\ \rho>0}} \frac{1}{\rho^\lambda} \|f\|^p_{L^p(\mathcal{B}_\rho(x)\cap\Omega)} < +\infty$$

where $\rho \in (0, \operatorname{diam}\Omega)$.

Can we define Morrey-type spaces over unbounded domain $\Omega$, i.e. $\operatorname{diam}\Omega = \infty$?

**Definition (Transirico, Troisi, Vitolo, 1995)**

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be unbounded. The Morrey-type space $M^{p,\lambda}(\Omega, d)$, $p \in [1, +\infty)$, $\lambda \in (0, n)$, $d > 0$ consists of $g \in L^p_{\mathrm{loc}}(\Omega)$ s.t.

$$\|g\|^p_{M^{p,\lambda}(\Omega,d)} = \sup_{\substack{\tau\in(0,d]\\ x\in\Omega}} \frac{1}{\tau^\lambda} \int_{\Omega\cap\mathcal{B}_\tau(x)} |g(y)|^p \, dy < +\infty \,.$$

Let $\Omega$ be an open subset of $\mathbb{R}^n$, $n \geq 2$

- $\forall\, d_1, d_2 \in \mathbb{R}_+ \;\exists\, c_1, c_2 \in \mathbb{R}_+$ :

$$c_1 \|g\|_{M^{p,\lambda}(\Omega,d_1)} \leq \|g\|_{M^{p,\lambda}(\Omega,d_2)} \leq c_2 \|g\|_{M^{p,\lambda}(\Omega,d_1)} .$$

- If $p \leq p_0$ and $\dfrac{\lambda - n}{p} \leq \dfrac{\lambda_0 - n}{p_0}$, then

$$M^{p_0,\lambda_0}(\Omega, d) \hookrightarrow M^{p,\lambda}(\Omega, d) .$$

- For each $\tau \in (0, d]$, $d > 1$ and $g \in M^{p,\lambda}(\Omega, d)$

$$\|g\|_{M^{p,\lambda}(\Omega,d)} \leq \|g\|_{M^{p,\lambda}(\Omega,\tau)} + d^{-\lambda/p}\|g\|_{M^{p,\lambda}(\Omega,d)} .$$

- If $\lambda = 0$ then $M^{p,0}(\Omega, d) \equiv M^{p}(\Omega, d)$ definite in **Transirico, Troisi, '87**.

$$\|g\|_{M^{p}(\Omega,d)} = \sup_{\substack{\tau \in (0,d] \\ x \in \Omega}} \|g\|_{L^{p}(\Omega \cap \mathcal{B}_\tau(x))} .$$

- If $\Omega \equiv \mathbb{R}^n$ then $L^{p,\lambda}(\mathbb{R}^n) \subset M^{p,\lambda}(\mathbb{R}^n, d)$.

- $VM^{p,\lambda}(\Omega, d)$ is the subspace of $M^{p,\lambda}(\Omega, d)$ for which

$$\lim_{d\to 0} \|g\|_{M^{p,\lambda}(\Omega,d)} = 0 \,.$$

- $\tilde{M}^{p,\lambda}(\Omega, d)$ is the closure of $L^{\infty}(\Omega)$ w.r.t. the norm in $M^{p,\lambda}(\Omega, d)$.
- $M_0^{p,\lambda}(\Omega, d)$ is the closure of $C_0^{\infty}(\Omega)$ w.r.t. the norm in $M^{p,\lambda}(\Omega, d)$.

The classical *Calderón-Zygmund decomposition* $(L^1, L^\infty)$ asserts that for any $f \in L^1$ and any $t > 0$ holds

$$f = f_t + (f - f_t), \qquad f_t \in L^\infty(\mathbb{R}^n),\ f - f_t \in L^1(\mathbb{R}^n)$$

where $\|f_t\|_{L^\infty(\mathbb{R}^n)} \le c(n)t.$
Analogous decomposition exists also for the functions from $\tilde{M}^{p,\lambda}(\Omega, d)$ and $M_0^{p,\lambda}(\Omega, d)$. In the spirit of CZD we write $g = g_h + (g - g_h),\ \forall h > 0$. Here $g_h$ is the good part which is more regular and $g - g_h$ is the bad part, whose norm can be controlled via the modulus of $g$ in the corresponding space.

Consider

$$\begin{cases} Lu = -D_j(a_{ij}D_iu + d_ju) + b_iD_iu + cu & \text{in } \Omega \\ x = 0 & \text{on } \partial\Omega \end{cases}$$

- $a_{ij} \in L^\infty(\Omega)$ and

$$a_{ij}(x)\xi_i\xi_j \geq \nu|\xi|^2 \quad \text{for a.a. } x \in \Omega,\ \forall\, \xi \in \mathbb{R}^n.$$

- $b_i, d_i \in M^{2r,n-2r}(\Omega, d)$, $c \in M^{r,n-2r}(\Omega, d)$ for some $r \in (1, n/2]$.

## Theorem (Transirico, Troisi, Vitolo '95)

*Let* $b_i - d_i \in M_0^{2r,n-2r}(\Omega, d)$ *and* $c - D_i(d_i) \geq \mu > 0$. *Then if* $u$ *is a solution of*

$$\begin{cases} Lu = f \in W^{-1,2}(\Omega, d) \ \text{ with unbounded } \ \Omega, \\ u \in W_0^{1,2}(\Omega, d) \end{cases}$$

*then it satisfies*

$$\|u\|_{W^{1,2}(\Omega,d)} \leq C\|f\|_{W^{-1,2}(\Omega,d)}\,.$$

OUR AIM: To extend the spaces $M^{p,\lambda}(\Omega, d)$ to spaces of Morrey type with a weight $\omega(\mathcal{B}_\tau(x)) : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ in open sets $\Omega \subseteq \mathbb{R}^n$ not necessarily bounded.

## Definition (Caso, D'Ambrosio, S., '17)

Let $\omega : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ be a measurable function. For $1 \le p < +\infty$ and $d \in \mathbb{R}_+$ fixed, we denote by $M^p_\omega(\Omega, d)$ the class of all functions $f \in L^p_{loc}(\bar{\Omega})$ such that

$$\|f\|_{M^p_\omega(\Omega,d)} = \sup_{\substack{x\in\Omega \\ \tau\in(0,d]}} \left( \frac{1}{\omega(x,\tau)} \int_{\Omega\cap\mathcal{B}_\tau(x)} |f(y)|^p \, dy \right)^{\frac{1}{p}} < +\infty$$

- If $\Omega \equiv \mathbb{R}^n$ then

$$\|f\|_{M^p_\omega(\mathbb{R}^n,d)} = \sup_{\substack{x\in\mathbb{R}^n \\ \tau\in(0,d]}} \left( \frac{1}{\omega(x,\tau)} \int_{\mathcal{B}_\tau(x)} |f(y)|^p \, dy \right)^{\frac{1}{p}} < +\infty$$

- $L^{p,\omega}(\mathbb{R}^n) \subset M^p_\omega(\mathbb{R}^n, d)$ .

Moreover, if $\omega(x,\tau) = \tau^\lambda$, with $0 < \lambda < n$, then

$$L^{p,\lambda}(\mathbb{R}^n) \subset M^p_\omega(\mathbb{R}^n, d) \, .$$

$W_1$) Doubling condition:

$$\frac{1}{C_\omega} \le \frac{\omega(x,s)}{\omega(x,r)} \le C_\omega \qquad \forall\, x \in \Omega, \quad r \le s \le 2r,$$

where $C_\omega \in \mathbb{R}_+$ is independent of $x$, $r$, $s$.

$W_2$) Monotonicity condition: If $\mathcal{B}_r(y) \subseteq \mathcal{B}_s(x)$ then

$$\omega(y,r) \le \omega(x,s) \quad \forall\, x,\, y \in \Omega \ \forall r, s \in \mathbb{R}_+$$

## Example

The following weights satisfy $W_1$ and $W_2$ :

- Let $u \in A_p$ be a Muckenhoupt weight with $1 < p < 1/\alpha$, $\alpha \in (0,1)$, then

$$\omega(x,\tau) = \Big( \int_{\mathcal{B}_\tau(x)} u(y)\, dy \Big)^{\alpha}, \forall\, (x,\tau) \in \mathbb{R}^n \times \mathbb{R}_+ \,.$$

- $\omega(x,\tau) = \Phi(\tau)$, where $\Phi$ is Young function satisfying the $\triangle_2$-condition: $\Phi(2\tau) \le k\Phi(\tau),\ \tau > 0$.
- $\omega(x,\tau) = \phi(\tau)$, where $\phi(\tau)$ is increasing function s.t. $\phi(\tau)/\tau$ is decreasing.

Under assumptions $W_1$ and $W_2$, making use of a diadic decomposition, we get the following result:

**Theorem**

*Let $p \in [1, +\infty)$ and $d_1, d_2$ be positive constants, then*

$$f \in M^p_\omega(\Omega, d_1) \quad \textit{iff} \quad f \in M^p_\omega(\Omega, d_2)$$

*and*

$$\|f\|_{M^p_\omega(\Omega,d_1)} \le \|f\|_{M^p_\omega(\Omega,d_2)} \le c\,\|f\|_{M^p_\omega(\Omega,d_1)}$$

$\forall\, d_1, d_2 > 0$, *where* $c \in \mathbb{R}_+$ *depends on* $n, p, C_\omega, d_1, d_2$.

$W_3)$ Suppose that for each $p \in [1, +\infty)$ holds

$$\lim_{d\to 0} \sup_{\substack{x\in\Omega \\ \tau\in(0,d]}} \frac{|\Omega(x,\tau)|}{\omega(x,\tau)} = D < +\infty,$$

which is equivalent to $\lim_{d\to 0} \|\chi_\Omega\|_{M^p_\omega(\Omega,d)} = D^{1/p} < +\infty.$

**Lemma**

*If $\omega$ satisfies $W_1$, $W_2$ and $W_3$, then*

$$M^p_\omega(\Omega, d) \subseteq M^q_\omega(\Omega, d) \qquad \forall\ 1 \leq q \leq p < +\infty.$$

*In addition $L^\infty(\Omega) \subset M^p_\omega(\Omega, d)$.*

## Definition

For $p \in [1, +\infty)$, we denote by $VM^p_\omega(\Omega, d)$ the subspace consisting of the functions $g \in M^p_\omega(\Omega, d)$ s.t.

$$\lim_{d \to 0} \|g\|_{M^p_\omega(\Omega,d)} = 0 \,.$$

Suppose that $\omega$ satisfies

$W_3'$) Uniform continuity: $\forall\, p \in [1, +\infty)$

$$\lim_{d \to 0} \sup_{\substack{x \in \Omega \\ \tau \in (0,d]}} \frac{|\Omega(x,\tau)|}{\omega(x,\tau)} = 0$$

which is equivalent to $\lim_{d \to 0} \|\chi_\Omega\|_{M^p_\omega(\Omega,d)} = 0.$

## Lemma

*If $\omega$ satisfies $W_1$, $W_2$ and $W_3$, then*

$$L^\infty(\Omega) \subset VM^p_\omega(\Omega, d).$$

*If we consider $W_3'$ instead of $W_3$, then*

$$M^p_\omega(\Omega, d) \subset VM^q_\omega(\Omega, d) \qquad \forall\ 1 \le q \le p < +\infty.$$

## Example

The weight $\omega = \phi(d)$ s.t.

$$\lim_{d\to 0} \phi(d) = 0, \qquad \lim_{d\to 0} \frac{\phi(d)}{d} = +\infty$$

verifies $W_3'$.

- Regularity of the boundary: $\Omega$ satisfies the (A)-condition, that is

$$\text{(A)} \qquad \sup_{\substack{x\in\Omega \\ \tau\in(0,d]}} \frac{|\mathcal{B}_\tau(x)|}{|\Omega(x,\tau)|} = A \; < \; +\infty\,.$$

The condition (A) implies the *external cone condition*

$$|\Omega(x,\tau)| \geq \frac{1}{A}\tau^n, \qquad \forall\, x \in \Omega,\; \tau \in (0,d].$$

Some properties of the spaces $VM^p_\omega(\Omega, d)$ :

## Lemma

*Let* (A), $W_1$, $W_2$ *and* $W_3$ *hold. If* $g \in VM^p_\omega(\Omega, d)$ *and* $\operatorname{supp} g \Subset \Omega$ *then*

$$\lim_{y \to 0} \|g(\cdot - y) - g(\cdot)\|_{M^p_\omega(\Omega,d)} = 0\,,$$

$$\lim_{h \to +\infty} \|J_h * g - g\|_{M^p_\omega(\Omega,d)} = 0\,,$$

*where* $(J_h)_{h\in\mathbb{N}}$ *is a sequence of mollifies in* $\mathbb{R}^n$ *.*

## Definition

Let $\omega$ satisfy $W_1$, $W_2$ and $W_3$. Denote by $\widetilde{M}^p_\omega(\Omega, d)$ the subspace of $M^p_\omega(\Omega, d)$ s.t. for each $E \subset \Omega$

$$\lim_{h\to+\infty}\left(\sup_{\|\chi_E\|_{M^p_\omega(\Omega,d)}<\frac{1}{h}} \|g\chi_E\|_{M^p_\omega(\Omega,d)}\right) = 0\,.$$

## Main properties

- The space $\widetilde{M}^p_\omega(\Omega, d)$ is the closure of $L^\infty(\Omega)$ w.r.t. the norm in $M^p_\omega(\Omega, d)$ .
- For all $1 \le q < p < +\infty$ the inclusion holds

$$M^p_\omega(\Omega, d) \subset \widetilde{M}^q_\omega(\Omega, d) .$$

- Let $\omega$ satisfy in addition $W_3'$, then

$$\widetilde{M}^p_\omega(\Omega, d) \subset VM^p_\omega(\Omega, d).$$

Define the cut off functions $\zeta_h \in C_0^\infty(\mathbb{R}^n)$ s.t.

$$\zeta_h(x) = \begin{cases} 1 & x \in \mathcal{B}_h(0) \\ 0 & x \notin \mathcal{B}_{2h}(0) \,. \end{cases}$$

**Definition (The space $\overset{\circ}{M}{}^p_\omega(\Omega, d)$)**

Let $w$ satisfies $W_1$, $W_2$ and $W_3'$. Then $g \in M^p_\omega(\Omega, d)$ belongs to $\overset{\circ}{M}{}^p_\omega(\Omega, d)$ iff

$$g \in \widetilde{M}^p_\omega(\Omega, d) \;\text{ and }\; \lim_{h \to +\infty} \|(1 - \zeta_h) g\|_{M^p_\omega(\Omega,d)} = 0 \,.$$

We can describe $\overset{\circ}{M}{}^p_\omega(\Omega, d)$ by means of the following density results.

**Lemma**

*A function $g \in M^p_\omega(\Omega, d)$ belongs to $\overset{\circ}{M}{}^p_\omega(\Omega, d)$ iff $g$ is in the closure of $C_0^\infty(\Omega)$ w.r.t. the norm in $M^p_\omega(\Omega, d)$.*

## *Embeddings between Morrey type subspaces*

- For all $p \in [1, +\infty)$ and $\omega$ satisfying $W_3$

$$\mathring{M}^p_\omega(\Omega, d) \subset \widetilde{M}^p_\omega(\Omega, d) \subset VM^p_\omega(\Omega, d) \subset M^p_\omega(\Omega, d)\,.$$

- For all $1 \leq p < q < \infty$

$$M^q_\omega(\Omega, d) \subset M^p_\omega(\Omega, d) \quad M^q_\omega(\Omega, d) \subset \widetilde{M}^p_\omega(\Omega, d)\,.$$

We want to construct decomposition for the functions belonging to $\widetilde{M}^p_\omega(\Omega, d)$ and $\overset{\circ}{M}{}^p_\omega(\Omega, d)$, in the spirit of the classical CZD.

## Modulus of continuity of $g$ in $\widetilde{M}^p_\omega(\Omega, d)$.

We define modulus continuity of $g$ in $\widetilde{M}^p_\omega(\Omega, d)$ as a map $\widetilde{\sigma}^p_\omega[g] : \mathbb{R}_+ \to \mathbb{R}_+$ such that for each measurable $E \subset \Omega$

$$\begin{cases} \displaystyle\sup_{\|\chi_E\|_{M^p_\omega(\Omega,d)} \leq \frac{1}{h}} \|g\,\chi_E\|_{M^p_\omega(\Omega,d)} \leq \widetilde{\sigma}^p_\omega[g](h)\,, \\ \displaystyle\lim_{h\to+\infty} \widetilde{\sigma}^p_\omega[g](h) = 0\,. \end{cases}$$

## Modulus of continuity in $\overset{\circ}{M}{}^{p}_{\omega}(\Omega, d)$.

We define modulus of continuity of $g$ in $\overset{\circ}{M}{}^{p}_{\omega}(\Omega, d)$ as a map $\overset{\circ}{\sigma}{}^{p}_{\omega}[g] : \mathbb{R}_+ \to \mathbb{R}_+$ such that for each measurable $E \subset \Omega$

$$\begin{cases} \sup\limits_{\|\chi_E\|_{M^p_\omega(\Omega,d)} \le \frac{1}{h}} \|g\,\chi_E\|_{M^p_\omega(\Omega,d)} + \|(1-\zeta_h)\,g\|_{M^p_\omega(\Omega,d)} \le \overset{\circ}{\sigma}{}^{p}_{\omega}[g](h), \\ \lim\limits_{h\to+\infty} \overset{\circ}{\sigma}{}^{p}_{\omega}[g](h) = 0\,. \end{cases}$$

## Theorem (Caso, D'Ambrosio, S., '17)

*Let* $g \in \widetilde{M}^p_\omega(\Omega, d)$, *then for any* $h > 0$ *we have* $g = (g - g_h) + g_h$, *where*

$$\begin{cases} \|g - g_h\|_{M^p_\omega(\Omega,d)} \leq \widetilde{\sigma}^p_\omega[g](h)\,, \\ \\ g_h \in L^\infty(\Omega)\,,\ \|g_h\|_{L^\infty(\Omega)} \leq\ h^{\frac{1}{p}}\|g\|_{M^p_\omega(\Omega,d)}\,. \end{cases}$$

## Proof.

For any $g \in \widetilde{M}^p_\omega(\Omega, d)$ consider the upper level sets

$$E_h = \{x \in \Omega : \ |g(x)| \geq h^{\frac{1}{p}} \|g\|_{M^p_\omega(\Omega,d)}\} .$$

Then for $x \in \Omega$ and $\tau \in (0, d]$

$$\frac{|E_h(x, \tau)|}{\omega(x, \tau)} \leq \frac{1}{h} .$$

Define the function $g_h = g\chi_{E_h}$, then $(1 - \chi_{E_h})g \in L^\infty(\Omega)$ while

$$\|g_h\|_{M^p_\omega(\Omega,d)} \leq \sup_{\|\chi_E\|_{M^p_\omega(\Omega,d)} \leq \frac{1}{h}} \|g_h\|_{M^p_\omega(\Omega,d)} \leq \widetilde{\sigma}^p_\omega[g](h) .$$

□

### Theorem

*Let* $g \in \overset{\circ}{M}{}^p_\omega(\Omega, d)$, *then for any* $h > 0$ *we have* $g = (g - g_h) + g_h$, *where*

$$\begin{cases} \|g - g_h\|_{M^p_\omega(\Omega,d)} \leq \overset{\circ}{\sigma}{}^p_\omega[g](h)\,, \\ |g_h| \leq \zeta_h\, h^{\frac{1}{p}} \|g\|_{M^p_\omega(\Omega,d)}\,. \end{cases}$$

## Proof.

Fix $g \in \overset{\circ}{M}{}^p_\omega(\Omega, d)$ and define the functions

$$g_h = \zeta_h(1 - \chi_{E_h})g\,, \text{ and } g - g_h = (1 - \zeta_h)g + \zeta_h\chi_{E_h}g\,.$$

Then

$$\begin{aligned}\|g - g_h\|_{M^p_\omega(\Omega,d)} &\leq \|(1 - \zeta_h)g\|_{M^p_\omega(\Omega,d)} + \|\chi_{E_h}g\|_{M^p_\omega(\Omega,d)} \\ &\leq \overset{\circ}{\sigma}{}^p_\omega[g](h)\end{aligned}$$

while by the definition of $E_h$

$$|g_h| \leq \zeta_h\, h^{\frac{1}{p}}\|g\|_{M^p_\omega(\Omega,d)}\,.$$

□

# Bibliography I


- Caso, L., P. Cavaliere, M. Transirico,
  *Existence results for elliptic equations*,
  J. Math. Anal. Appl., **274** (2) (2002), 554–563.

- Caso, L., R. D'Ambrosio, L. Softova,
  *A kind of generalized Morrey spaces over unbounded domains,* (submitted).

- Chiarenza, F., M. Franciosi,
  *A generalization of a theorem by C.Miranda*,
  Ann. Mat. Pura Appl., **161** (1992), 285–297.

- Morrey Jr., C.B.,
  *On the solutions of quasi-linear elliptic partial differential equations,*
  Trans. Amer. Math. Soc., **43** (1938), 126–166.

# Bibliography II


Morrey Jr., C.B.,
Second order elliptic equations in several variables and Hölder continuity,
Math. Zeitschr., **72** (1959), 146–164.

Nakai, E.,
*Hardy-Littlewood maximal operator, singular integral operators and the Riesz potentials on generalized Morrey spaces,*
Math. Nachr., **166** (1994), 95–103.

# Bibliography III


Transirico, M., M. Troisi,
Second order elliptic equations with discontinuous coefficients and of variational type in open unbounded domains,
Boll. Unione Mat. Ital., VII. Ser., B **2**, No.2 (1988), 385–398.

Transirico, M., M. Troisi, A. Vitolo,
*Spaces of Morrey type and elliptic equations in divergence form on unbounded domains*,
Boll. Unione Mat. Ital., **9** (1995), 153–174.

Zorko, C.T.,
*Morrey spaces,*
Proc. Amer. Math. Soc., **98** (1986), 586–592.


Doğum Günüm Kutlu Olsun, Vagif!