Dynamics and Chaotic Behaviors of The Iteration Process of The Functions

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Introduction

In this presentation, first of all let us briefly discuss the historical development of dynamical systems. Historical beginnings of dynamical systems have began with Newton discovering the differential calculus. In this period, mathematical analysis has implemented many techniques and applications, but in this great development, there are some missing points. These missing points include the linear systems of mathematics that developed at that time. However, real life has include many non-linear situations. **Newton's n-body problem** is the origin point of dynamical systems. This problem involves both the motion of the n-body itself and the effect on n-tuple system. The n-body problem can be solved by using the differential equation for $n = 1$ and $n = 2$, but the solution for $n = 3$ can not be obtained. The main reason why this problem can not be solved for $n = 3$ is that the system is in a chaotic structure and the chaos is not yet defined.

In the historical process, the second stage of the development of dynamical systems was started by King Oscar II who announced in 1860 that there will be a reward for scientists who can solve the n-body problem. In this step, J. Henri **Poincaré** came in sight. Poincaré has proved that there is no solution of such a differential equation for $n = 3$ using topology and geometry. However, he hasn't noticed the chaos that exists in the problem. Between 1920 and 1930, **P. Fatou and G. Juli** [1] revealed chaotic structure on the Julia clusters by using complex analytical transformations, but they couldn't classify real chaos since there was no computer support. On the same dates, G.D. Birkhoff has studied iterative processes as the simplest way of understanding the dynamics behaviors of differential equations based on Poincaré's quantitative perspective.

A major development on dynamical systems occurred in 1960. The American mathematician **S. Smale** [2] took into account Poincaré's stable and unstable passes depending on the iteration processes, he has shown the chaotic movement as an example by using symbolic dynamics. At the same time, the American meteorologist **E.N. Lorentz** [3] examplifyed the types of chaos obtained by Poincaré using computer. Lorentz described dynamical systems as dependence sensitive to initial conditions and showed that long-term weather forecasts were not possible as a result of his experiments.

Since 1970, dynamical systems and chaos have been studied in different branches of science. Physicist **M. Feigenbaum** [4] has shown that each chaotic system can reach a constant (i.e., chaos constant). In the field of medicine, pulsation is displayed by the second order function, known as logistic equation, and its dynamics and chaos were studied, [5].

What is the dynamical system?

The simplest answer to this question is the result that we get by repeatedly pressing the exponential function key after pressing a number key of a calculator. The resulting mathematical formula in the calculator is $x, e^x, e^{e^x}, e^{e^{e^x}}, \ldots$.This formula will give error after a certain step. So this sequence goes to infinite. Given the above representation as a more general structure, suppose that we have given x_0 a initial point and f a function, then we get compound function sequence as

$$x_{0,} f(x_0), f(f(x_0)), f(f(f(x_0))), \ldots = x_{0,} f(x_0), f^2(x_0), f^3(x_0), \ldots. \quad (1)$$

This compound functions sequence are called the dynamics of the initial value x_0. The function iteration is an effort to control the position of function which is subject to x_0 initial point in the process by the mathematical analysis except probability issue. At this stage, we examine the dynamics of the function iterations by some example. Our first example has related to finance.

Example

Let's assume that we invested **deposit** of 1000 \$ in a bank with 10% annual interest rate. How can we check the status of this deposit after 15 years?

Solution: Let A_0 be a starting point and $A_0 = 1000$ \$. We get

$$\begin{aligned}
\text{1st year deposit:} \quad & A_1 = A_0 + (0,1)A_0 = (1,1)A_0, \\
\text{2st year deposit:} \quad & A_2 = A_1 + (0,1)A_1 = (1,1)A_1 = (1,1)^2 A_0, \\
& \vdots \\
\text{nst year deposit:} \quad & A_n = A_{n-1} + (0,1)A_{n-1} = (1,1)A_{n-1} = (1,1)^n A_0.
\end{aligned}$$

After this step, we define a suitable function

$$F(x) = (1,1)x \tag{2}$$

on cluster $\{A_0, A_1, A_2, \ldots, \ A_n\}$.

Example

By using (2) and its compound of the functions, we obtain that

$$\begin{aligned} A_0 &= F^0(A_0) = A_0 \\ A_1 &= F(A_0) = (1,1)A_0 \\ A_2 &= F^2(A_1) = (1,1)^2 A_0 \\ &\vdots \\ A_n &= F^n(A_{n-1}) = (1,1)^n A_0, \end{aligned}$$

where $F^0(A_0) = A_0$. And we find 15*st* year deposit situation

$$A_{15} = F^{15}(A_{14}) = (1,1)^{15} 10^3.$$

Now, we will talk about briefly use of dynamical system in population control the following example.

Example (Logical equation of population growth)

Let us suppose that there is a population supported and balanced by the environment. If the population exceed the predicted value then it will be balanced by **disease, food shortage** and related **deaths**. We will give the equation of change in the form of

$$P_{n+1} = \lambda P_n(1 - P_n),$$

where λ is a constant depending on environmental conditions, $0 < \lambda \leq 4$ and $0 \leq P_n \leq 1$. To calculate the dynamics that will arise from the increase and decrease of the population, we will give the nonlinear second order equation

$$F_\lambda(x) = \lambda x(1 - x),$$

called **logistic function**. At the end of the talk, we will show that the value of λ is **chaos** in this function.

The basic problems in dynamic systems are interpretation of the movements of (1) presentation. There are many mathematical techniques for interpreted the presentation. We will give some of them within subject and we will show what kind of interpretations are obtained from these techniques. Firstly, we define the **orbit** of a dynamical system.

Definition (Orbit)

The set of points $x_{0,} f(x_0), f^2(x_0), f^3(x_0), \ldots$ is forward orbit of initial point x_0. If f is a **homeomorphism**, then $f^n(x_0)$ and $f^{-n}(x_0)$ is a orbit

for all $n \in \mathbb{Z}$.

There exists different techniques for classification of orbits. One of these is the fixed points that will be described below.

Definition (Fixed Point)

Let f be a function. A arbitrary point x is said to be a fixed point such that

$$f(x) = x.$$

The orbit generated by the fixed point is a fixed orbit. The fixed points of the function $f(x) = x^2$ are the values $x_1 = 0$ and $x_2 = 1$ of solution to the equation $x^2 - x$. An orbit with this initial point is $\left(1, f(1), f^2(1), f^3(1)\right) = (1, 1, 1, \ldots)$ constant orbit. It can be repeated for the same iteration value.

Another method used to determine another orbit from a fixed point is the period.

Definition (Period)

Let f be a function, x be a arbitrary point and for $\forall n \in \mathbb{Z}$, we say that x is said to be n-period periodical point of function f such that

$$f^n(x) = x.$$

Also the smallest number n is called the **prime period** of x.

Although the fixed points of the $f(x) = x^2 - 1$ is $\left(\frac{1 \mp \sqrt{5}}{2}\right)$, the values obtained from the solution of the equation $x^2 - 2x - x = 0$ are 0 and 1, except for the fixed points. Thus, 0 and 1 are periodic points with prime period of 2.

Types and Behaviors of Periodic Points

Let $f : I \to I$ be a function and $I \subset \mathbb{R}$. Suppose that fixed point of f is $s \in I$ such that $f(s) = s$. In this case, there exist the following situations for the point:

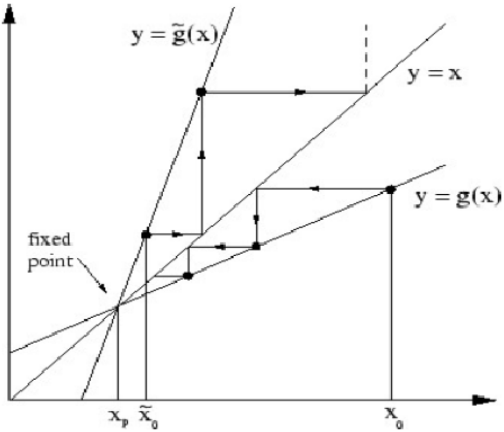
1. If $|f'(s)| < 1$ then point s is **attractive**,
2. If $|f'(s)| = 1$ then point s is **neutral**,
3. If $|f'(s)| > 1$ then point s is **repellent.**

In a similar way, for a periodic point p with prime period $n \geq 2$

1. If $|(f^n)(p)| < 1$ then point p is **attractive**,
2. If $|(f^n)(s)| = 1$ then point p is **neutral,**
3. If $|(f^n)(s)| > 1$ then point p is **repellent.**

Graphic Analysis

An important method for evaluating functional iterations is graphical analysis. This method helps to evaluate the dynamics of a function geometrically. To determine the dynamics of a function g, the line $y = x$ is drawn and an initial point x_0 is selected. From this point, the curve of the g function is reached and $y = x$ is returned. Thus, the orbit of an initial point x_0 is interpreted geometrically and information about its type is obtained. In the following way the initial points for $y = g(x)$ and $y = \widetilde{g}(x)$ are x_0 and $\widetilde{x_0}$, respectively. x_p is a fixed point. As can be seen, it is understood that x_p is the **attractive point** for $y = g(x)$ and a **repulsive point** for $y = \widetilde{g}(x)$.


y = g̃(x)
y = x
y = g(x)
fixed point
x_p
$\tilde{x}_0$
x_0

BIFURCATIONS

Now we will consider a family of second order functions $q_c(x) = x^2 + c$, let $c \in \mathbb{R}$ be a constant. The simple appearance of these functions is misleading because their dynamics are incredibly complex. Indeed, the dynamics obtained for certain values "c" are not yet defined. Herein, c is a parameter and a new dynamic is obtained for each value. As the value of c changes, let we start the following process to examine the dynamics of q_c. To find the fixed points of family $q_c(x) = x^2 + c$, we need to find the roots of equation $x^2 + c = x$. The roots we will get from this equation as follow

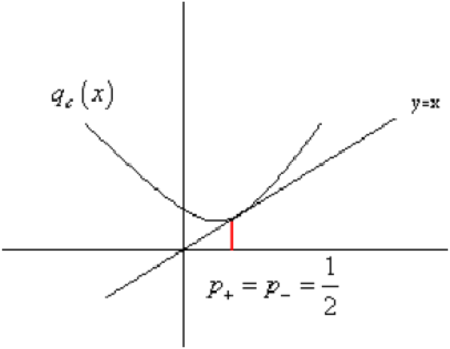
$$
\begin{aligned}
p_+(c) &= \frac{1}{2}\left(1 + \sqrt{1 - 4c}\right) \\
p_-(c) &= \frac{1}{2}\left(1 - \sqrt{1 - 4c}\right).
\end{aligned}
$$

For these roots to be real, it must be $c \leq \frac{1}{4}$. For this reason we are working on the real right as the table below:

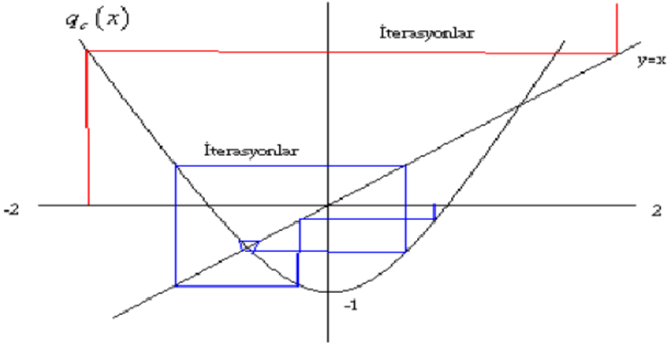
- For $c > \frac{1}{4}$, q_c has no fixed point.
- For $c = \frac{1}{4}$, q_c has a unique fixed point and $p_+ = p_- = \frac{1}{2}$.
- For $c < \frac{1}{4}$, q_c has two different real fixed point such that $p_+(c) > p_-(c)$.

In the case of $c > \frac{1}{4}$, the dynamics are simple. Because the graph of q_c is a parabola whose arms are open upward and does not intersect with $y = x$. All the orbits go to infinity.

The fixed point of equation $q_c(x) = x^2 + c$ are $q'_c(p_+) = 1 + \sqrt{1-4c}$ and $q'_c(p_-) = 1 - \sqrt{1-4c}$, so there is a unique fixed point for $c = \frac{1}{4}$ and it is neutral.



But if $q_c'(p_+) > 1$ for $c < \frac{1}{4}$, then the fixed point $p_+(c)$ is always a repellent. On the other hand, when $0 < q_c'(p_-) < 1$, the fixed point $p_+(c)$ is always attractive. Indeed, since $|q_c'(p_-)| < 1$ necessary and sufficient condition is -$\frac{3}{4} < c < \frac{1}{4}$.



The point to take into account in here is that the **first bifurcation** occurs when $q_c(x) = x^2 + c$ falls below the value c of $\frac{1}{4}$. In other words, the fixed point $c = \frac{1}{4}$ is divided into two. The bifurcation is called the **tangent bifurcation**. The point to be emphasized that when $x_0 > p_+$ and $x_0 < p_-$ the orbits of x_0 go to infinity. So all the interesting dynamics in this bifurcation take place in the range of $[-p_+, p_+]$.

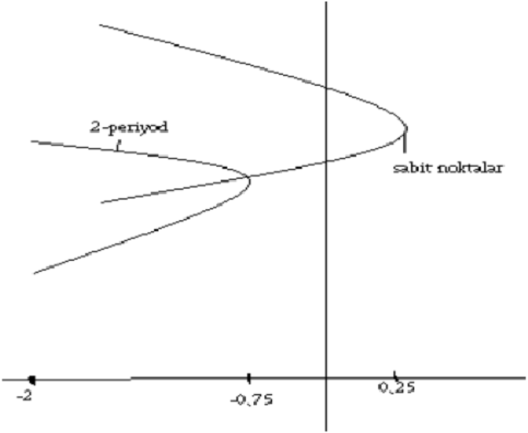
What happens when c falls below value $-\frac{3}{4}$?

The answer to this question is possible with the solution of the equation $q_c^2(x) = x$. Since $q_c^2(x) - x = \left(x^2 - x + c\right)\left(x^2 + x + c + 1\right)$, the roots of the equation $\left(x^2 - x + c\right)$ are fixed points of $q_c^2(x)$, since it is fixed points of $q_c(x)$. So the roots of the equation $\left(x^2 + x + c + 1\right)$ are only fixed points of $q_c^2(x)$. These **roots** are as follows:

$$s_{\pm}(c) = \frac{1}{2}\left(-1 \pm \sqrt{-4c-3}\right).$$

It is necessary and sufficient condition $c \leq \frac{3}{4}$ for the above roots to be real. Thus, a new bifurcation type arises. In this branching, if $c \leq -\frac{3}{4}$ then the fixed point p_- turns into repellent from attraction and $s_{\pm}(c)$ becomes 2-turn cycle. If $c = -\frac{3}{4}$, then $s_+ = s_- = p_-$. After this step, we can draw a graphic in family of $q_c(x) = x^2 + c$ that will branch to fixed points and real value c.

Let us give values c on x-axis and fixed point obtained for variable values on y-axis. This graphic is as follows:



General Properties of the Second Bifurcation

- For $-\frac{3}{4} < c < \frac{1}{4}$, q_c has **attractive** fixed point at p_-, but not 2-periodic point.
- For $c = -\frac{3}{4}$, q_c has non-periodic $p_- = s_\pm$ and **neutral** fixed point.
- For $-\frac{5}{4} < c < \frac{3}{4}$, q_c has **repellent** fixed points at $p_\pm$ and 2-periodic **attractive** fixed point at $s_\pm$.

Functions connected to a parameter have branching characteristics in general. Of course, the function here must be **continuous** and **differentiable** according to the parameter it depends on. We can give examples of such functions; $f_\lambda(x) = x^2 + \lambda$ (second order equations), $f_\lambda(x) = \lambda x(1-x)$ (logistic family) and $e_\lambda(x) = e^x + \lambda$ function classes. An important bifurcation of these function classes according to the λ parameter is the **saddle-node bifurcation**, which will be described below.

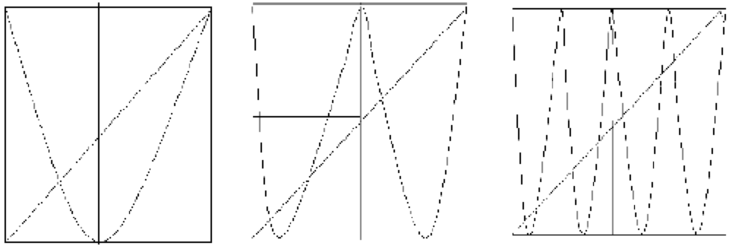
Definition

Let I be open interval and $\varepsilon > 0$. The **saddle-node bifurcation** which family of f_λ functions with 1-parameter has the following properties:
1) For $\lambda_0 - \varepsilon < \lambda < \lambda_0$, f_λ has not fixed point on open interval I.
2) For $\lambda_0 = \lambda$, f_λ has **neutral** fixed point on open interval I.
3) For $\lambda_0 < \lambda < \lambda_0 + \varepsilon$, f_λ has **attractive** and **repellent** two fixed points on open interval I.

Importance of parametre in bifurcation is seen obviously. In the iteration of the function, the cyclic and period are occur depends on the bifurcation at the fixed point. These cycle and period lead us to the concept of chaos and chaos transition which another important feature of functional iterations.

$c = -2$ **in family** $q_c(x) = x^2 + c$

We have mentioned that the dynamics of family $q_c(x) = x^2 + c$ are realized in the $[-p_+, p_+]$. Now, we will focus on $I = [-2, 2]$ by taking $c = -2$ and $p_+ = 2$. All the iterations of q stays in a square with 2 edge in this range. There is a relationship among the fixed points obtained from the iterations of $q_{-2}(x)$ as follows.



We give the relation by using hole numbers of the curves in the boxes. A graphical analysis of iterations $q_{-2}(x)$, $q^2_{-2}(x)$ and $q^3_{-2}(x)$ is shown at $I = [-2, 2]$ in above.If the curve of $q_{-2}(x)$ has one pit, then it has 2 fixed points, if curve of $q^2_{-2}(x)$ has 2 pits , then it has 4 fixed points, if the curve of $q^3_{-2}(x)$ has 4 pits , it has 8 fixed points. Also if this iteration is continued, there will be n holes in the curve of $q^n_{-2}(x)$. Therefore we will obtain 2^n fixed points. In the light of this information we can give the following theorem.

Theorem

The function $q_{-2}(x)$ has n-periodical 2^n periodic points in the interval $2 \leq x \leq 2$.

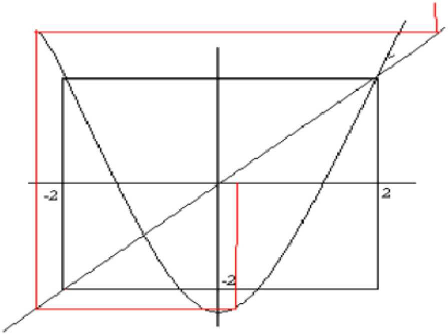
$c < -2$ in family $q_c(x) = x^2 + c$

The dynamics obtained the family $q_c(x) = x^2 + c$ for $c < -2$ will lead us to interesting results. Consider the square box having corners $(-p_+, -p_+), (-p_+, p_+), (p_+, -p_+), (p_+, p_+)$ in the plane. As is known, the orbit of the $x_0 \notin [-p_+, p_+]$ initial point goes to infinity. That is, $q^n(x_0) \to \infty$ as $n \to \infty$. As can be seen, the graph of q_c overflows out of the square box when $c < -2$. The intersection points of the curve of q_c when an open interval containing zero moves out of the box, the orbits formed as the result of the first iteration of q_c go out of this open interval and go to infinity. Let's express this open interval as A_1. It is clear that $A_1 \subset I$. Let

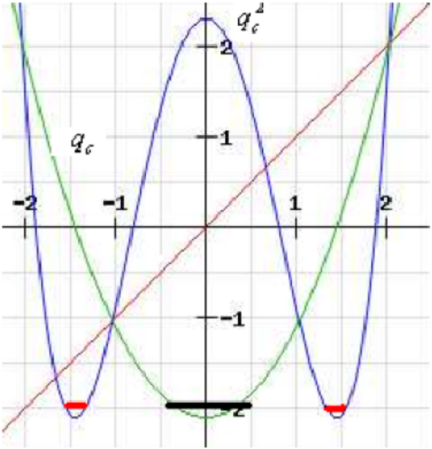
$$\Lambda = \{x \in I : q_c^n(x) \in I, n \in \mathbb{N}\},$$

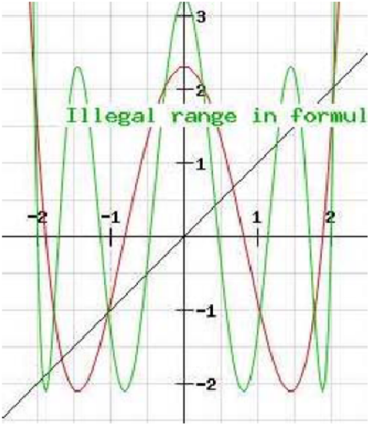
denote the set of points that not depart from A_1 by the iteration of the function q_c of I.

The graph of $q_c(x) = x^2 + c$ for $c < -2$ in the interval $2 \leq x \leq 2$ is given below:



The above graphs belong to functions q_c^2 and q_c^3, respectively. When we continue the above process, let we have shown that A_2 is the set of points that its orbits move away from I and go to infinity under q_c^2 , and A_3 is the set of points that its orbits go to infinity under q_c^3.





When we continue this process, the cluster A_n contains exactly 2^{n-1} open interval, and these open interval include the points moving away from the interval I. In this word

$$\Lambda = \{x \in I : q_c^n(x) \in I, n \in \mathbb{N}\},$$

is the closed set remaining after all open intervals in I are removed. This expression indicates that the $\Lambda = \{x \in I : q_c^n(x) \in I, n \in \mathbb{N}\}$ is the **Cantor set**.

So far we have examined the iterations of the $q_c(x) = x^2 + c$ functions and the results of the iterations.

FUNCTION ITERATIONS AND CHAOS

In the historical process, chaos was noticed by **Edward Lorenz** [3] firstly. In 1960, he had done to try in order to find approximate numerical solution equation including the model of atmosphere of earth consist of 13 function classes. He had changed initial value of these functions classes. When he came back to check new situation, he was shocked. He tought that the computers were broken. After checking all computers, he enterenced new valueds and then he recognized that small change on input brought about incredible changing on output. What Edward Lorenz faced was the notion of chaotic behavior. **Meteorological equations** meaning that a small change in the initial conditions (for example, a temperature difference of one degree) could lead to a sunrise one month later or a very rainy day. Even in a small change in the initial conditions, it can cause large changes which will make it impossible to be recognized. Even if the difference between the two sets of values taken for the initial conditions grows, the irregular situation also occurs quickly. Now, we can give mathematically definition of chaos.

Definition

Let X be metric spaces and $F : X \longrightarrow X$ be continuous mapping. A dynamical system F is **chaotic** if it holds the following three conditions:

1. Periodic point for F are **dense**
2. F is **transitive**
3. F **depends sensitively** on initial conditions.

In this step, to understand chaotic behavior we will give an example related to ecology by using logistics equation. Let's suppose that there is an insect society with a population of $500,000$ which gives special environmental conditions. Let the population in first year be x_1 million, that is, fixed population is $x_1 = 0,5$.

According to **logistics equation**, we have the following steps

$$\begin{aligned} x_2 &= 2x_1\left(1-x_1\right) \\ x_3 &= 2x_2\left(1-x_2\right) \\ &\vdots \\ x_{n+1} &= 2x_n\left(1-x_n\right). \end{aligned}$$

In this equation, by taking initial value between 0 and 1, we can get the population. Let $x_1 = 0,8$. Then we get the following sequence:

$$\begin{array}{l} 0,8 \\ 0,32 \\ 0,4352 \\ 0,4916 \\ 0,49986 \\ 0,499999 \\ \vdots \end{array}$$

It is easy to see that this sequence approaches to fixed value $(x_1 = 0,5)$ of the population at $6th$ years, and then it converges to $x_1 = 0,5$ as $n \to \infty$. This point is **stable** fixed point of logistic equation, therefore it is attractor for all ortbits. Now, let us take another initial value close to $x = 0$ which is the other fixed point. Then we can observe that ortbits close the $x = 0$ approach to stable fixed point because the point 0 is reppel fixed point. Since constant λ depend on environmental conditions, general form of **logistic equation** can be given by

$$f_\lambda(x) = \lambda x (1 - x)$$

Now let us choose 0.6875 million in place of initial population 0.5 million. Then the logistic equation is the following form

$$f_{3.2}(x) = 3.2x(1 - x)$$

For initial value $x_0 = 0.6875$. After n-iteration process $f^n(x_0) = 0.6875$. It is not efficient of environmental conditions. Basically, if $\left| f_{3.2}'(x) \right| > 1$, then initial populations is not attraction. Indeed, when taking initial values close to $x_0 = 0.6875$, the elements in orbits made oscillate between points $x_0 = 0,799$ and $x_1 = 0,513$.

Excel formulation is $3,2 * A1 * (1 - A1)$ such that $A1 = 0,799$

$0,799$	$0,5139168$
$0,5139168$	$0,799380233$
$0,799380233$	$0,513188724$
$0,513188724$	$0,799443384$
$0,799443384$	$0,513067711$
$0,513067711$	$0,799453552$
$0,799453552$	$0,513048225$
$0,513048225$	$0,79945518$
$0,79945518$	$0,513045104$
$0,513045104$	$0,799455441$
$0,799455441$	$0,513044605$

The above iteration sequence has 2-cyclic period. The reason of this event, since fixed point $x = 0.6875$ is **unstable**, it has oscillating orbits between $0,513044605$ and $0,799380233$ which are population of insects. If we take fixed population $x = 0,71428$, logistic function can be given by $f_{3.5}(x) = 3.5x(1-x)$. For this function, by taking initial value close to point $x = 0,71428$, let us iterate it.

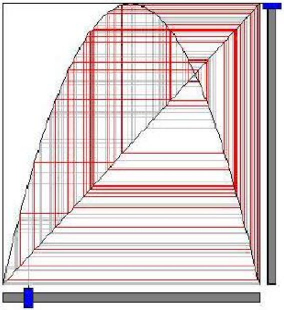
We obtain 4-cyclic period as $a, b, c, d, a, b, c, d, a, b, c, d, \ldots$such that

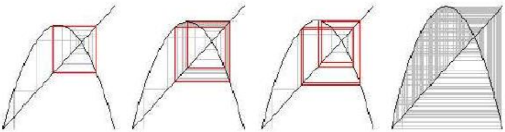
$$a = 0,382819; b = 0,826940; c = 0,50088; d = 0,874997.$$

Since $\left| f'_{3,5}(0,71428) \right| = 1$, the fixed point is **neutral**. Therefore it is **unstable**.

If the population in the season "a" is less than necessary, this causes a population strike in the next "b" season and a decrease in the season "c". Finally, the population "c" is less than necessary and triggers the last increase in season "d". As long as the stationary point is neutral, this cyclic structure never breaks down and remains stable. If $\lambda > 3.6$ is taken, the iterations of the function tend to take the form of a force like $8, 16, 32, 64, \ldots$ instead of the fixed point. This is the basic approach that leads us to **chaos.**

The logistic equation for $2 \leq \lambda \leq 4$ shows the cycles and chaos as follows.





At this stage, let's examine the behavior of two insect populations whose logistic equation is $\lambda \geq 3.9$ and whose starting values are very close to each other. Begin with two insect communities whose starting values are $x_1 = 0.100$ and $x_1 = 0.101$.

Let us examine the following table which gives the situation of these two insect communities during the first 14 generations.

Generations	**Community I**	**Community II**
1	0.100	0.101
2	0.351	0.354
3	0.888	0.892
4	0.387	0.376
5	0.925	0.915
6	0.271	0.304
7	0.771	0.825
8	0.690	0.562
9	0.835	0.960
10	0.538	0.150
11	0.969	0.497
12	0.116	0.975
13	0.399	0.095
14	0.935	0.336

As you can see, the proximity between the two populations has completely disappeared when the $10th$ season comes. This descriptive equation we use to trace this mysterious transition from the regular state to the chaos has no magic feature. Numerous other simple links can lead to this kind of behavior. It is this uniformity that makes researching the transition to chaos so intriguing. We can explain the **common feature** as follows: When a λ parameter in the equation slowly approaches the critical limit L, the solutions go into chaos under the **"period folding"** process, which is consistently twice as high as the oscillation period instead of a stable fixed point. This gives us the ability to distinctively distinguish between different or powerful chaotic behaviors. The approach to the critical limit, L from the regular side, doubling the number of repetitions is the most striking representation of general behavior.

Finally, we give chaos constant as follows: Let

$$d_n = \frac{\lambda_{n+1} - \lambda_n}{\lambda_{n+2} - \lambda_{n+1}}$$

and

$$\lim_{n \to \infty} d_n = d = 4,6692016,$$

this limit is called a **chaos constant.**

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Thank you for listening to me...