Integral Operators of Harmonic Analysis in Local Morrey-Lorentz Spaces

Ayhan SERBETCI
Ankara University, Department of Mathematics, Ankara, Turkey

Dedicated to 60th birthday of professor Vagif S. GULİYEV
July 10-13, KIRSEHIR/TURKEY

Introduction

The theory of boundedness of classical operators of harmonic analysis from one weighted Lebesgue space to another one is by now well studied. These results have good applications in the theory of partial differential equations. However, it should be noted that in the theory of partial differential equations, along with weighted Lebesgue spaces, general Morrey-type spaces also play an important role.

In a series of papers by Guliyev, Aykol, Kucukaslan and Serbetci (2013, 2016, 2016 and 2017) the local Morrey-Lorentz spaces $M^{loc}_{p,q;\lambda}(\mathbb{R}^n)$ have been introduced and the basic properties of these spaces have been given, and the boundedness of the Hilbert transform H, the Hardy-Littlewood maximal operator M and the Calderón-Zygmund operators T, and Riesz potential I_α on the local Morrey-Lorentz spaces $M^{loc}_{p,q;\lambda}(\mathbb{R}^n)$ has been extensively studied, respectively.

Introduction

This talk is dedicated to the results obtained by the author jointly with his co-authors V.S. Guliyev, C. Aykol, A. Kucukaslan. We will give the basic properties of the local Morrey-Lorentz spaces $M^{loc}_{p,q;\lambda}(\mathbb{R}^n)$ and the boundedness of classical operators of harmonic analysis, such as the Hardy-Littlewood maximal operator M, Hilbert transform H, the Calderón-Zygmund operators T, and Riesz potential I_α will be given on the local Morrey-Lorentz spaces $M^{loc}_{p,q;\lambda}(\mathbb{R}^n)$.

Local Morrey-Lorentz spaces $M^{loc}_{p,q;\lambda}(\mathbb{R}^n)$ are generalizations of Lorentz spaces $L_{p,q}(\mathbb{R}^n)$ such that for the case $\lambda = 0$ we get the space $L_{p,q}(\mathbb{R}^n)$.

Introduction

Some type of Morrey-Lorentz spaces were studied by some authors (see Mingione 2010, Ho 2014 and Ragusa 2012). Mingione [18] defined the Lorentz-Morrey spaces $\mathcal{L}_{p,q;\lambda}(\mathbb{R}^n)$ by following:

Definition

(Mingione 2010) The Lorentz-Morrey spaces $\mathcal{L}_{p,q;\lambda}(\mathbb{R}^n)$ is the set of all measurable functions f on $\mathbb{R}^n$: for $1 \leq p < \infty$, $0 < q < \infty$, and $0 \leq \lambda \leq n$, $f \in \mathcal{L}_{p,q;\lambda}(\mathbb{R}^n)$ if and only if

$$\|f\|_{\mathcal{L}_{p,q;\lambda}} = \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|\chi_{B(x,r)} f\|_{L_{p,q}} < \infty.$$

Introduction

Mingione studied the boundedness of the restricted fractional maximal operator $M_{\beta,B}$:

$$M_{\beta,B}f(x) = \sup_{B(x,r)\subset B} |B(x,r)|^{\frac{\beta}{n}-1} \int_{B(x,r)} |f(y)|dy, \ x \in \mathbb{R}^n$$

in the restricted Lorentz-Morrey spaces $\mathcal{L}_{p,q;\lambda}(B)$, where B is any ball. Mingione derives a general non-linear version, extending a priori estimates and regularity results for possibly degenerate non-linear elliptic problems to the various spaces of Lorentz and Lorentz-Morrey type considered in (Adams, 1981, 1982 and 1996).

Ragusa [21] defined the Lorentz-Morrey spaces $L_{p,q;\lambda}(\mathbb{R}^n)$ and studied some embeddings between these spaces.

Definition

(Ragusa, 2012) The Lorentz-Morrey spaces $L_{p,q;\lambda}(\mathbb{R}^n)$ is the set of all measurable functions f on $\mathbb{R}^n$: for $1 \leq p < \infty$, $0 < q < \infty$, and $0 \leq \lambda \leq n$, if and only if

$$\|f\|_{L_{p,q;\lambda}} = \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{q}} \|\chi_{B(x,r)} f\|_{L_{p,q}} < \infty.$$

Note that the spaces $\mathcal{L}_{p,q;\lambda}(\mathbb{R}^n)$ and $L_{p,q;\lambda\frac{q}{p}}(\mathbb{R}^n)$ defined by Mingione and Ragusa respectively, coincide, thus

$$\mathcal{L}_{p,q;\lambda}(\mathbb{R}^n) = L_{p,q;\lambda\frac{q}{p}}(\mathbb{R}^n).$$

We shall use the following notation. For a Lebesgue measurable set $E \subset \mathbb{R}^n$ and $0 < p \leq \infty$, $L_p(E)$ is the standard Lebesgue space of all functions f Lebesgue measurable on E for which

$$\|f\|_{L_p(E)} := \left(\int_E |f(y)|^p dy \right)^{\frac{1}{p}} < \infty,$$

if $0 < p < \infty$ and

$$\|f\|_{L_\infty(E)} := \sup\{\alpha : |\{y \in E : |f(y)| \geq \alpha\}| > 0\} < \infty,$$

if $p = \infty$. Also, for an open set $E \subset \mathbb{R}^n$, $L_p^{loc}(E)$ is the set of all functions f such that $f \in L_p(K)$ for any compact $K \subset E$. If $E = \mathbb{R}^n$, then, for brevity, we write L_p for $L_p(\mathbb{R}^n)$ and L_p^{loc} for $L_p^{loc}(\mathbb{R}^n)$.

Preliminaries

The same convention refers to the case of weak Lebesgue spaces $WL_p(E)$, the space of all functions f Lebesgue measurable on E for which

$$\|f\|_{WL_p(E)} := \sup_{0<t\leq|E|} t^{1/p} f^*(t), \quad 1 \leq p < \infty$$

and

$$\|f\|_{WL_\infty} \equiv \|f\|_{L_\infty}, \quad p = \infty.$$

Here $|E|$ is the Lebesgue measure of E, and f^* denotes the right continuous non-increasing rearrangement of f:

$$f^*(t) := inf\left\{\alpha > 0 : \mu_f(\alpha) \leq t\right\}, \ \forall t \in (0, \infty),$$

where

$$\mu_f(\alpha) := |\{y \in \mathbb{R} : |f(y)| > \alpha\}|$$

is the distribution function of f.

Preliminaries

In the following we give the local Morrey spaces $LM_{p,\lambda}(0,\infty)$ (see, Alvarez, Lakey, Guzman-Partida 2000, N. Samko 2009- 2012).

Definition

Let $1 \le p < \infty$ and $0 \le \lambda \le 1$. We denote by $LM_{p,\lambda} \equiv LM_{p,\lambda}(0,\infty)$ the local Morrey space, the space of all functions $\varphi \in L_p^{\rm loc}(0,\infty)$ with finite quasinorm

$$\|\varphi\|_{LM_{p,\lambda}} = \sup_{r>0} r^{-\frac{\lambda}{p}} \|\varphi\|_{L_p(0,r)}.$$

Also by $WLM_{p,\lambda} \equiv WLM_{p,\lambda}(0,\infty)$ we denote the weak local Morrey space of all functions $\varphi \in WL_p^{\rm loc}(0,\infty)$ for which

$$\|\varphi\|_{WLM_{p,\lambda}} = \sup_{r>0} r^{-\frac{\lambda}{p}} \|\varphi\|_{WL_p(0,r)} < \infty.$$

Lorentz spaces are introduced by Lorentz in 1950. These spaces are quasi-Banach spaces and generalizations of the more familiar L_p spaces, also they are appear to be useful in the general interpolation theory.

Definition

The Lorentz space $L_{p,q} \equiv L_{p,q}(\mathbb{R}^n)$, $0 < p, q \leq \infty$, is defined as the set of all measurable functions f on $\mathbb{R}^n$ with finite quasi-norm

$$\|f\|_{L_{p,q}} := \|\tau^{\frac{1}{p}-\frac{1}{q}} f^*(\tau)\|_{L_q(0,\infty)}.$$

The functional $\|\cdot\|_{L_{p,q}}$ is a norm if and only if either $1 \leq q \leq p$ or $p = q = \infty$. If $p = q = \infty$, then the space $L_{\infty,\infty}(\mathbb{R}^n)$ is denoted by $L_\infty(\mathbb{R}^n)$. Clearly $L_{p,p} \equiv L_p$ and $L_{p,\infty} \equiv WL_p$.

We denote by $\mathfrak{M}(\mathbb{R}^n)$ be the set of all extended real valued measurable functions on $\mathbb{R}^n$ and $\mathfrak{M}^+(0,\infty)$ the set of all non-negative measurable functions on $(0,\infty)$.

Definition

Let $0 < p \le \infty$ and $\psi \in \mathfrak{M}^+(0,\infty)$. We denote by $\Lambda_{p,\psi}(\mathbb{R}^n)$ the classical Lorentz spaces, the spaces of all measurable functions with finite quasinorm

$$\Lambda_{p,\psi}(\mathbb{R}^n) := \{f \in \mathfrak{M}(\mathbb{R}^n) : \|f\|_{\Lambda_{p,\psi}} := \|\psi f^*\|_{L_p(0,\infty)}\}.$$

Therefore, for $\psi(t) = t^{\frac{1}{p}-\frac{1}{q}}$, $0 < p, q \le \infty$ we get the spaces $\Lambda_{p,t^{\frac{1}{p}-\frac{1}{q}}}(\mathbb{R}^n)$ are equal with norm coincide to the Lorentz spaces $L_{p,q}(\mathbb{R}^n)$.

Definition

Let $f \in L_1^{loc}(\mathbb{R}^n)$. The space of functions with bounded mean oscillation, $BMO \equiv BMO(\mathbb{R}^n)$, consists of those functions f for which

$$\|f\|_{BMO} := \sup_{B \subset \mathbb{R}^n} \frac{1}{|B|} \int_B |f - f_B| dx$$

is finite, where the supremum is taken over open balls $B \subset \mathbb{R}^n$ and

$$f_B = \frac{1}{|B|} \int_B f dx.$$

Definition

(Aykol, Guliyev, Serbetci, 2013) Let $0 < p, q \leq \infty$ and $0 \leq \lambda \leq 1$. We denote by $M^{loc}_{p,q;\lambda} \equiv M^{loc}_{p,q;\lambda}(\mathbb{R}^n)$ the local Morrey-Lorentz spaces, the spaces of all measurable functions with finite quasinorm

$$\|f\|_{M^{loc}_{p,q;\lambda}} := \sup_{r>0} r^{-\frac{\lambda}{q}} \|t^{\frac{1}{p}-\frac{1}{q}} f^*(t)\|_{L_q(0,r)}.$$

If $\lambda < 0$ or $\lambda > 1$, then $M^{loc}_{p,q;\lambda}(\mathbb{R}^n) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

Lemma

1) In the limiting case $\lambda = 0$ the space $M^{loc}_{p,q;0}(\mathbb{R}^n)$ is the Lorentz space $L_{p,q}(\mathbb{R}^n)$.
That is $M^{loc}_{p,q;0}(\mathbb{R}^n) = L_{p,q}(\mathbb{R}^n)$, $0 < p, q \le \infty$ and

$$\|f\|_{M^{loc}_{p,q;0}} = \|f\|_{L_{p,q}} \equiv \|\tau^{\frac{1}{p}-\frac{1}{q}} f^*(\tau)\|_{L_q(0,\infty)}.$$

2) In the limiting case $\lambda = 1$ the space $M^{loc}_{p,q;1}(\mathbb{R}^n)$ is the classical Lorentz space $\Lambda_{\infty, t^{\frac{1}{p}-\frac{1}{q}}}(\mathbb{R}^n)$. That is

$$M^{loc}_{p,q;1}(\mathbb{R}^n) = \Lambda_{\infty, t^{\frac{1}{p}-\frac{1}{q}}}(\mathbb{R}^n), \; 0 < p, q \le \infty$$

and

$$\|f\|_{M^{loc}_{p,q;1}} = \|f\|_{\Lambda_{\infty, t^{\frac{1}{p}-\frac{1}{q}}}} \equiv \|\tau^{\frac{1}{p}-\frac{1}{q}} f^*(\tau)\|_{L_\infty(0,\infty)}.$$

Lemma

If f be a measurable function, then $\|f\|_{M^{loc}_{p,q;\lambda}}$ is a quasi-norm on $M^{loc}_{p,q;\lambda}(\mathbb{R}^n)$. If $1 \le q \le p$ or $p = q = \infty$, then the functional $\|f\|_{M^{loc}_{p,q;\lambda}}$ is a norm.

Lemma

If $p=\infty$, then we have $M^{loc}_{\infty,q;\lambda}=\Theta$ for any $0<q<\infty$.

Proof.

Assume that $M^{loc}_{\infty,q;\lambda}\neq\Theta$. Then there exists a nonzero function $f\in M^{loc}_{\infty,q;\lambda}$ which means that there exists $c>0$ and a positive measurable set A such that $|f(x)|\geq c$ for all $x\in A$. Then

$$\begin{aligned}\|f\|_{M^{loc}_{\infty,q;\lambda}} &= \sup_{r>0} r^{-\frac{\lambda}{q}}\|t^{-\frac{1}{q}}f^*(t)\|_{L_q(0,r)}\\ &\geq \sup_{r>0} r^{-\frac{\lambda}{q}}\|t^{-\frac{1}{q}}(f\chi_A)^*(t)\|_{L_q(0,r)}\\ &\geq c\sup_{r>0} r^{-\frac{\lambda}{q}}\|t^{-\frac{1}{q}}\|_{L_q(0,\min\{|A|,r\})}=\infty.\end{aligned}$$

□

Basic properties of local Morrey-Lorentz type spaces

Lemma

Let $0 < q \le p < \infty$, $\frac{1}{r} = \frac{1}{p} - \frac{\lambda}{q}$ *and* $0 < \lambda \le \frac{q}{p}$. *Then*

$$M^{loc}_{p,q;\lambda}(\mathbb{R}^n) \equiv WL_r(\mathbb{R}^n)$$

and

$$\left(\frac{q}{p}\right)^{-\frac{1}{q}} \|f\|_{WL_r} \le \|f\|_{M^{loc}_{p,q;\lambda}} \le \lambda^{-\frac{1}{q}} \|f\|_{WL_r}, \quad f \in WL_r(\mathbb{R}^n).$$

Corollary

Let $0 < q \le p < \infty$. *Then*

$$M^{loc}_{p,q;\frac{q}{p}}(\mathbb{R}^n) \equiv L_\infty(\mathbb{R}^n).$$

Lemma

Let $0 < p, q < \infty$ *and* $0 \leq \lambda \leq 1$. *Then*

$$M^{loc}_{p,q;\lambda}(\mathbb{R}^n) \hookrightarrow L_{p,q;n\lambda}(\mathbb{R}^n)$$

and for $f \in M^{loc}_{p,q;\lambda}(\mathbb{R}^n)$

$$\|f\|_{L_{p,q;n\lambda}} \lesssim \|f\|_{M^{loc}_{p,q;\lambda}}.$$

For $x \in \mathbb{R}^n$ and $r > 0$, we denote by $B(x, r)$ the open ball centered at x of radius r, and by $B(x, r)^c$ denote its complement. Let $|B(x, r)|$ be the Lebesgue measure of the ball $B(x, r)$. Therefore $|B(x, r)| = \omega_n r^n$, ω_n denotes the volume of unit sphere S^{n-1} in $\mathbb{R}^n$. For $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$, the Hardy-Littlewood maximal function Mf of f is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x, r)|} \int_{B(x,r)} |f(y)| dy, \ x \in \mathbb{R}^n.$$

It is well known that for the classical Hardy-Littlewood maximal operator the rearrangement inequality

$$cf^{**}(t) \le (Mf)^{*}(t) \le Cf^{**}(t), \;\; t \in (0, \infty) \qquad (1)$$

holds (Bennett and Sharpley, 1988), where the positive constants c, C are independent of f and t, and

$$f^{**}(t) = \frac{1}{t}\int_0^t f^{*}(s)ds.$$

We will use the following Hardy operator to obtain the boundedness of the maximal operator M, Hilbert operator H, Calderon-Zygmund operator T and maximal Calderon-Zygmund operator $\mathcal{T}$ in the local Morrey-Lorentz spaces $M^{loc}_{p,q;\lambda}$.

Definition

(N. Samko, 2012) Let f be a measurable function on $(0,\infty)$ and β be a real number. The weighted Hardy operators A_β and $\mathcal{A}_\beta$ with power weights acting on f are defined by

$$A_\beta f(t) = t^{\beta-1}\int_0^t \frac{f(s)}{s^\beta}ds, \quad \mathcal{A}_\beta f(t) = t^\beta \int_t^\infty \frac{f(s)}{s^{\beta+1}}ds.$$

The following theorem was proved by N. Samko, 2012.

Theorem

Theorem A. *Let* $\beta \in \mathbb{R}$, $0 \le \lambda < 1$ *and* $1 \le q < \infty$. *If* $\beta < \frac{\lambda}{q} + \frac{1}{q'}$ *and* $\beta > \frac{\lambda}{q} - \frac{1}{q}$, *then the operators* A_β *and* $\mathcal{A}_\beta$ *are bounded on the local Morrey space* $LM_{q,\lambda}(0,\infty)$, *respectively.*

The following theorem was proved by Aykol, Guliyev, Kucukaslan, Serbetci, 2016 .

Theorem

Theorem B. *Let* $\beta \in \mathbb{R}$, $0 \le \lambda < 1$ *and* $1 < q < \infty$. *If* $\beta = \frac{\lambda}{q} + \frac{1}{q'}$ *and* $\beta = \frac{\lambda}{q} - \frac{1}{q}$, *then the operators* A_β *and* $\mathcal{A}_\beta$ *are bounded from the lokal Morrey space* $LM_{q,\lambda}(0,\infty)$ *to the weak local Morrey space* $WLM_{q,\lambda}(0,\infty)$, *respectively.*

In the following theorem we prove the boundedness of maximal operator M on the local Morrey-Lorentz spaces $M^{loc}_{p,q;\lambda}(\mathbb{R}^n)$.

Theorem

(Guliyev, Aykol, Kucukaslan, Serbetci, 2016)
Let $1 \leq q \leq \infty$, $0 \leq \lambda < 1$ *and* $\frac{q}{q+\lambda} \leq p \leq \infty$.
(i) If $\frac{q}{q+\lambda} < p < \infty$, *then the maximal operator* M *is bounded on the local Morrey-Lorentz space* $M^{loc}_{p,q;\lambda}$.
(ii) If $p = \frac{q}{q+\lambda}$, *then the operator* M *is bounded from* $M^{loc}_{p,q;\lambda}$ *to the weak local Morrey- Lorentz space* $WM^{loc}_{p,q;\lambda}$.
(iii) If $p = q = \infty$, *then the operator* M *is bounded on* $L_\infty(\mathbb{R}^n)$.

Proof.

(i) Suppose $\frac{q}{q+\lambda} < p < \infty$ and $f \in M^{loc}_{p,q;\lambda}$. From the definition in local Morrey-Lorentz spaces and inequality (1) we get

$$\begin{aligned}\|Mf\|_{M^{loc}_{p,q;\lambda}} &= \sup_{r>0} r^{-\frac{\lambda}{q}} \|t^{\frac{1}{p}-\frac{1}{q}} f^*(t)\|_{L_q(0,r)} \\ &\leq C \sup_{r>0} r^{-\frac{\lambda}{q}} \left\| t^{\frac{1}{p}-\frac{1}{q}-1} \int_0^t f^*(s)ds \right\|_{L_q(0,r)} \\ &= C \left\| A_{(\frac{1}{p}-\frac{1}{q})} g \right\|_{LM_{q,\lambda}(0,\infty)},\end{aligned}$$

where $g(t) = t^{\frac{1}{p}-\frac{1}{q}} f^*(t)$. Since $\frac{1}{p} - \frac{\lambda}{q} < 1$, for $\beta = \frac{1}{p} - \frac{1}{q}$ the inequality $\beta < \frac{\lambda}{q} + \frac{1}{q'}$ holds. From the boundedness of Hardy operator we get

$$\|A_{(\frac{1}{p}-\frac{1}{q})} g\|_{LM_{q,\lambda}(0,\infty)} \leq C\|g\|_{LM_{q,\lambda}(0,\infty)} = C\|f\|_{M^{loc}_{p,q;\lambda}}.$$

Therefore we obtain the boundedness of M in $M^{loc}_{p,q;\lambda}$. □

Proof.

(ii) For the limiting case $p = \frac{q}{q+\lambda}$ suppose $f \in M^{loc}_{p,q;\lambda}$. We have

$$\|Mf\|_{WM^{loc}_{\frac{q}{q+\lambda},q;\lambda}} \leq C \sup_{r>0} r^{-\frac{\lambda}{q}} \left\| t^{\frac{\lambda-1}{q}} \int_0^t f^*(s)ds \right\|_{WL_q(0,t)}$$
$$= C\|A_\beta h\|_{WLM_{q,\lambda}(0,\infty)},$$

where $\beta = 1 + \frac{\lambda-1}{q}$ and $h(t) = t^{1+\frac{\lambda-1}{q}} f^*(t)$. From the boundedness of Hardy operator we get

$$\|A_\beta h\|_{WLM_{q,\lambda}(0,\infty)} \leq C\|h\|_{LM_{q,\lambda}(0,\infty)} = C\|f\|_{M^{loc}_{\frac{q}{q+\lambda},q;\lambda}}.$$

Then we obtain the boundedness of the operator M from the space $M^{loc}_{\frac{q}{q+\lambda},q;\lambda}$ to the weak space $WM^{loc}_{\frac{q}{q+\lambda},q;\lambda}$. □

Proof.

(iii) In the limiting case $p = \infty$, suppose $f \in M^{loc}_{\infty,q;\lambda}$. Since the space $M^{loc}_{\infty,q;\lambda}$ is trivial for any $0 < q < \infty$, we must consider the case $q = \infty$. Since the operator M is bounded on L_∞ we get the statement.

□

Let f be a locally integrable function on $\mathbb{R}$. The Hilbert transform Hf of f is defined by the principal-value integral

$$Hf(x) = p.v.\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(y)}{x-y}dy$$

provided it exists almost everywhere. The modified Hilbert transform $\widetilde{H}$ is defined as

$$\widetilde{H}f(x) := \lim_{\varepsilon\to 0^+}\int_{|x-y|>\varepsilon}\left[\frac{1}{x-y}+\frac{\chi(y)}{y}\right]f(y)dy,$$

where $\chi(y)$ is the characteristic function of $|y|>1$. The boundedness of the modified Hilbert transform $\widetilde{H}$ from the space L_∞ to the BMO space was proved by B. Muckenhoupt and R.L. Wheeden in 1976.

Hilbert transform

For each measurable function f on $(0, \infty)$ and each $t > 0$, the following operator

$$(Sf)(t) = \int_0^\infty \min\,(1, \frac{s}{t}) f(s) \frac{ds}{s}$$
$$= \frac{1}{t} \int_0^t f(s) ds + \int_t^\infty f(s) \frac{ds}{s}$$

was defined by A.P. Calderon (Calderon 1966). It is clear that S is linear. For the aim, its importance based on the fact that it dominates the maximal Hilbert transform.

Theorem

Theorem C. *(Bennett, Sharpley, 1988) Let* $f \in L_1^{loc}(\mathbb{R})$ *and suppose*

$$(Sf^*)(1) = \int_0^1 f^*(s)ds + \int_1^\infty f^*(s)\frac{ds}{s} < \infty. \tag{2}$$

Then

$$(\mathcal{H}f)^*(t) \leq CS(f^*)(t), \ 0 < t < \infty, \tag{3}$$

where C *is a constant independent of* f *and* t*.*

Theorem

Theorem D. *Let* $f \in L_1^{loc}(\mathbb{R})$ *and* f *satisfies* (2). *Then the Hilbert transform* $Hf(x)$, $x \in \mathbb{R}$ *exists almost everywhere. Furthermore,*

$$(Hf)^*(t) \leq CS(f^*)(t), \; 0 < t < \infty, \tag{4}$$

where C *is a constant independent of* f *and* t.

The following is a well known theorem about Hilbert transform.

Theorem

Theorem E. *Let* $0 < q \leq \infty$ *and* $1 \leq p \leq \infty$.
(i) *If* $1 < p < \infty$ $1 \leq q \leq \infty$, *then the operator* H *is bounded in the Lorentz space* $L_{p,q}$.
(ii) *If* $p = 1$ *and* $0 < q \leq 1$, *then the operator* H *is bounded from* $L_{1,q}$ *to the space* WL_1.
(iii) *If* $p = \infty$, *then the modified Hilbert operator* $\widetilde{H}$ *is bounded from* L_∞ *to* BMO.

Hilbert transform

The following theorem is the main result of our presentation, in which we get the analogue of Theorem E for the boundedness of the Hilbert transform in the local Morrey-Lorentz spaces $M^{loc}_{p,q;\lambda}$.

Theorem

(Aykol, Guliyev, Kucukaslan, Serbetci, 2016)
Suppose that $f \in M^{loc}_{p,q;\lambda}(\mathbb{R})$ *satisfies the equation* (2), $0 < q \le \infty$, $0 \le \lambda < 1$ *and* $\frac{q}{q+\lambda} \le p \le \frac{q}{\lambda}$. *Then the Hilbert transform Hf exists almost everywhere. Furthermore,*
(*i*) *If* $\frac{q}{q+\lambda} < p < \frac{q}{\lambda}$, $1 \le q < \infty$, *then the operator* H *is bounded in the local Morrey-Lorentz space* $M^{loc}_{p,q;\lambda}$.
(*ii*) *If* $p = \frac{q}{q+\lambda}$, $1 < q < \infty$, *then the operator* H *is bounded from* $M^{loc}_{p,q;\lambda}$ *to the weak space* $WM^{loc}_{p,q;\lambda}$.
(*iii*) *If* $p = \frac{q}{\lambda}$, $0 < q \le \infty$, *then the modified Hilbert operator* $\widetilde{H}$ *is bounded from* $M^{loc}_{p,q;\lambda}$ *to BMO.*

Proof.

Since f satisfies (2), from Theorem D the Hilbert transform $Hf(x)$, $x \in \mathbb{R}$ exists almost everywhere.

(i) Suppose that $1 \le q < \infty$, $0 \le \lambda < 1$, $\frac{q}{q+\lambda} < p < \frac{q}{\lambda}$ and $f \in M^{loc}_{p,q;\lambda}$. We have

$$\|Hf\|_{M^{loc}_{p,q,\lambda}} = \sup_{r>0} r^{-\frac{\lambda}{q}} \|t^{\frac{1}{p}-\frac{1}{q}}(Hf)^*(t)\|_{L_q(0,r)}$$

$$\le C \sup_{r>0} r^{-\frac{\lambda}{q}} \|t^{\frac{1}{p}-\frac{1}{q}} \left(\frac{1}{t}\int_0^t f^*(s)ds + \int_t^\infty \frac{f^*(s)}{s} ds\right) \|_{L_q(0,r)}$$

$$\le C\Big(\sup_{r>0} r^{-\frac{\lambda}{q}} \|t^{\frac{1}{p}-\frac{1}{q}-1} \int_0^t f^*(s)ds\|_{L_q(0,r)} + \sup_{r>0} r^{-\frac{\lambda}{q}} \|t^{\frac{1}{p}-\frac{1}{q}} \int_t^\infty \frac{f^*(s)}{s} ds\|$$

$$= I_1 + I_2.$$

□

Proof.

Let us estimate I_1:

$$I_1 = C \sup_{r>0} r^{-\frac{\lambda}{q}} \| t^{\frac{1}{p}-\frac{1}{q}-1} \int_0^t f^*(s)ds \|_{L_q(0,r)}$$
$$= C \| A_{(\frac{1}{p}-\frac{1}{q})} g \|_{LM_{q,\lambda}(0,\infty)},$$

where $g(t) = t^{\frac{1}{p}-\frac{1}{q}} f^*(t)$.
Since $\frac{1}{p} - \frac{\lambda}{q} < 1$, for $\beta = \frac{1}{p} - \frac{1}{q}$ the inequality $\beta < \frac{\lambda}{q} + \frac{1}{q'}$ holds. From Theorem A the operator A_β is bounded in the local Morrey spaces $LM_{q,\lambda}(0,\infty)$. Then,

$$I_1 \lesssim \| A_{(\frac{1}{p}-\frac{1}{q})} g \|_{L_{q,\lambda}(0,\infty)} \lesssim \| g \|_{L_{q,\lambda}(0,\infty)}$$
$$= \sup_{r>0} r^{-\frac{\lambda}{q}} \| t^{\frac{1}{p}-\frac{1}{q}} f^*(t) \|_{L_q(0,r)} = \| f \|_{M^{loc}_{p,q;\lambda}}. \quad (5)$$

□

Proof.

Now we consider I_2 :

$$I_2 = C \sup_{r>0} r^{-\frac{\lambda}{q}} \| t^{\frac{1}{p}-\frac{1}{q}} \int_t^\infty \frac{f^*(s)}{s} ds \|_{L_q(0,r)}$$
$$= C \| \mathcal{A}_{(\frac{1}{p}-\frac{1}{q})} g \|_{LM_{q,\lambda}(0,\infty)},$$

where, again, $g(t) = t^{\frac{1}{p}-\frac{1}{q}} f^*(t)$. Since $\frac{1}{p} - \frac{\lambda}{q} > 0$, for $\beta = \frac{1}{p} - \frac{1}{q}$ the inequality $\beta > \frac{\lambda}{q} - \frac{1}{q}$ holds. From Theorem B the operator $\mathcal{A}_\beta$ is bounded in the local Morrey spaces $LM_{q,\lambda}(0,\infty)$. Then,

$$I_2 \lesssim \| \mathcal{A}_{(\frac{1}{p}-\frac{1}{q})} g \|_{LM_{q,\lambda}(0,\infty)} \lesssim \| g \|_{LM_{q,\lambda}(0,\infty)}$$
$$= \sup_{r>0} r^{-\frac{\lambda}{q}} \| t^{\frac{1}{p}-\frac{1}{q}} f^*(t) \|_{L_q(0,r)} = \| f \|_{M^{loc}_{p,q;\lambda}}. \tag{6}$$

From the inequalities (5) and (6) we obtain the boundedness of the operator H in $M^{loc}_{p,q;\lambda}$.

Proof.

(ii) For the limiting case $p = \frac{q}{q+\lambda}$, $1 < q < \infty$, suppose $f \in M^{loc}_{p,q;\lambda}$. From the definition of norm in weak local Morrey-Lorentz spaces and by using the inequality (4) and Minkowski's inequality we get

$$\|Hf\|_{WM^{loc}_{\frac{q}{q+\lambda},q;\lambda}} = \sup_{r>0} r^{-\frac{\lambda}{q}} \|t^{\frac{q+\lambda}{q}-\frac{1}{q}} (Hf)^*(t)\|_{WL_q(0,r)}$$

$$= \sup_{r>0} r^{-\frac{\lambda}{q}} \|t^{1+\frac{\lambda-1}{q}} (Hf)^*(t)\|_{WL_q(0,r)}$$

$$\leq C \sup_{r>0} r^{-\frac{\lambda}{q}} \|t^{1+\frac{\lambda-1}{q}} \left(\frac{1}{t} \int_0^t f^*(s)ds + \int_t^\infty \frac{f^*(s)}{s} ds \right) \|_{WL_q(0,r)}$$

$$\leq C \sup_{r>0} r^{-\frac{\lambda}{q}} \|t^{\frac{\lambda-1}{q}} \int_0^t f^*(s)ds \|_{WL_q(0,r)}$$

$$+ C \sup_{r>0} r^{-\frac{\lambda}{q}} \|t^{1+\frac{\lambda-1}{q}} \int_t^\infty \frac{f^*(s)}{s} ds \|_{WL_q(0,r)} = N_1 + N_2.$$

□

Proof.

Let us estimate N_1:

$$N_1 = C \sup_{r>0} r^{-\frac{\lambda}{q}} \| t^{\frac{\lambda-1}{q}} \int_0^t f^*(s)ds \|_{WL_q(0,r)}$$
$$= C \|A_\beta h\|_{WLM_{q,\lambda}(0,\infty)},$$

where $\beta = 1 + \frac{\lambda-1}{q}$ and $h(t) = t^{1+\frac{\lambda-1}{q}} f^*(t)$. Therefore we get from Theorem B

$$N_1 \lesssim \|A_\beta h\|_{WLM_{q,\lambda}(0,\infty)} \lesssim \|h\|_{LM_{q,\lambda}(0,\infty)}$$
$$= \sup_{r>0} r^{-\frac{\lambda}{q}} \|t^{1+\frac{\lambda-1}{q}} f^*(t)\|_{L_q(0,r)}$$
$$= \|f\|_{M^{loc}_{\frac{q}{q+\lambda},q;\lambda}}. \quad (7)$$

□

Proof.

Now we consider N_2 :

$$N_2 = C \sup_{r>0} r^{-\frac{\lambda}{q}} \| t^{1+\frac{\lambda-1}{q}} \int_t^{\infty} \frac{f^*(s)}{s} ds \|_{WL_q(0,r)}$$
$$= C \| \mathcal{A}_\beta h \|_{WL_{q,\lambda}(0,\infty)},$$

where, again, $\beta = 1 + \frac{\lambda-1}{q}$ and $h(t) = t^{1+\frac{\lambda-1}{q}} f^*(t)$. From Lemma 2 (ii) the operator $\mathcal{A}_\beta$ is bounded from the Morrey spaces $L_{q,\lambda}(0,\infty)$ to $WL_{q,\lambda}(0,\infty)$. Then,

$$N_2 \lesssim \| \mathcal{A}_\beta h \|_{WL_{q,\lambda}(0,\infty)} \lesssim \| h \|_{L_{q,\lambda}(0,\infty)}$$
$$= \sup_{r>0} r^{-\frac{\lambda}{q}} \| t^{1+\frac{\lambda-1}{q}} f^*(t) \|_{L_q(0,r)} = \| f \|_{M^{loc}_{\frac{q}{q+\lambda},q;\lambda}}. \tag{8}$$

From the inequalities (7) and (8) we obtain the boundedness of the operator H from $M^{loc}_{\frac{q}{q+\lambda},q;\lambda}$ to $WM^{loc}_{\frac{q}{q+\lambda},q;\lambda}$.

Proof.

(iii) For the limiting case $p = \frac{q}{\lambda}$, $0 < q \leq \infty$, it will be convenient to use the modified Hilbert transform $\tilde{H}$ instead of H. The reason for using $\tilde{H}f$ is that it may exist while Hf may not exist (Dynkin, 1991).

Since $\widetilde{H}$ is bounded from L_∞ to BMO, the inequality

$$\|\widetilde{H}f\|_{BMO} \leq C\|f\|_{L_\infty} \equiv \|f\|_{WL_\infty}$$

holds (Muckenhoupt, Wheeden, 1976; Dynkin, 1991).

Then we get

$$\|\widetilde{H}f\|_{BMO} \leq C\|f\|_{M^{loc}_{\frac{q}{\lambda},q;\lambda}},$$

which proves that the modified Hilbert transform $\widetilde{H}$ is bounded from $M^{loc}_{\frac{q}{\lambda},q;\lambda}$ to BMO. □

Calderon-Zygmund operator

Suppose that $K \in L_1^{loc}(\mathbb{R}^n \setminus \{0\})$ and satisfies the following conditions:

$$(i)|K(x)| \leq \frac{C}{|x|^n}, \; x \in \mathbb{R}^n \setminus \{0\},$$

$$(ii) \int_{r_1<|x|<r_2} K(x)dx = 0, \;\; 0 < r_1 < r_2,$$

$$(iii)\, |K(x-y) - K(x)| \leq C\frac{|y|}{|x|^{n+1}} \;\; for \;\; 2|y| \leq |x|.$$

Then K is called the Calderon-Zygmund kernel, where C is a constant independent of x and y. Set

$$T_\varepsilon f(x) = \int_{B(x,\varepsilon)^c} K(x-y)f(y)dy.$$

We define the Calderon- Zygmund singular integral associated to K as

$$Tf(x) = (K * f)(x) = \lim_{\varepsilon \to 0} T_\varepsilon f(x).$$

and the maximal singular integral by

$$\mathcal{T} f(x) = \sup_{\epsilon > 0} \left| T_\varepsilon f(x) \right|.$$

The Calderon-Zygmund operator T extends to the whole space L_p, $1 \leq p < \infty$, by continuity. In the case $p = \infty$ we need a renormalization of T (see [Dyn'kin]). For this reason let us choose a point $x_0 \in \mathbb{R}^n$ and let $f \in L_\infty$. Set

$$T^0 f(x) = T(f\chi_{2B})(x) - T(f\chi_{2B})(x_0) + \int_{\complement_{B(x,r)}} [K(x-y) - K(x_0-y)] f(y) d$$

where $x_0 \in B(x,r)$. If $f \in L_p(\mathbb{R}^n)$, $p < \infty$, then obviously

$$T^0 f(x) = T(f)(x) - T(f)(x_0).$$

Theorem

Theorem F. *(Bennett, Sharpley, 1988) If* $f \in L_1^{loc}(\mathbb{R})$ *and suppose the equation* (2) *holds, then*

$$(\mathcal{T}f)^*(t) \leq CS(f^*)(t),\ 0 < t < \infty, \tag{9}$$

where C *is a constant independent of* f *and* t.

Theorem

Theorem G. *Let $f \in L_1^{loc}(\mathbb{R})$ and f satisfies (2). Then the Calderon-Zygmund operator T exists almost everywhere. Furthermore,*

$$(Tf)^*(t) \le CS(f^*)(t),\ 0 < t < \infty, \tag{10}$$

where C is a constant independent of f and t.

Calderon-Zygmund operator

The following theorem is the main result of our presentation, in which we get the boundedness of the Calderon-Zygmund operator in the local Morrey-Lorentz spaces $M^{loc}_{p,q;\lambda}$.

Theorem

Suppose that $f \in M^{loc}_{p,q;\lambda}$, $1 \le q \le \infty$, $0 \le \lambda < 1$, $\frac{q}{q+\lambda} \le p \le \frac{q}{\lambda}$ and the inequality (2) holds, then the Calderon-Zygmund integral $Tf(x)$ exists almost everywhere. Furthermore,
(i) If $1 \le q < \infty$, $\frac{q}{q+\lambda} < p < \frac{q}{\lambda}$, then the Calderon-Zygmund operator T is bounded on the local Morrey-Lorentz space $M^{loc}_{p,q;\lambda}$.
(ii) If $1 < q < \infty$, $p = \frac{q}{q+\lambda}$, then the operator T is bounded from $M^{loc}_{p,q;\lambda}$ to the weak local Morrey-Lorentz space $WM^{loc}_{p,q;\lambda}$.
(iii) If $1 \le q \le \infty$, $p = \frac{q}{\lambda}$, then the operator T^0 is bounded from $M^{loc}_{p,q;\lambda}$ to BMO.

Proof.

Let $1 \leq q \leq \infty$, $0 \leq \lambda < 1$ and $\frac{q}{q+\lambda} \leq p \leq \frac{q}{\lambda}$. Since f satisfies (2), by Theorem B the Calderon Zygmund operator Tf exists almost everywhere.

(i) Suppose that $1 \leq q < \infty$, $0 \leq \lambda < 1$, $\frac{q}{q+\lambda} < p < \frac{q}{\lambda}$ and $f \in M^{loc}_{p,q;\lambda}$. From the definition of norm in local Morrey-Lorentz spaces, by using the inequality (10) and Minkowski's inequality we get □

Proof.

$$\|Tf\|_{M^{loc}_{p,q,\lambda}} = \sup_{r>0} r^{-\frac{\lambda}{q}} \|t^{\frac{1}{p}-\frac{1}{q}} (Tf)^*(t)\|_{L_q(0,r)}$$
$$\leq C \sup_{r>0} r^{-\frac{\lambda}{q}} \left\| t^{\frac{1}{p}-\frac{1}{q}} \left(\frac{1}{t} \int_0^t f^*(s)ds + \int_t^\infty \frac{f^*(s)}{s} ds \right) \right\|_{L_q(0,r)}$$
$$\leq C \left(\sup_{r>0} r^{-\frac{\lambda}{q}} \left\| t^{\frac{1}{p}-\frac{1}{q}-1} \int_0^t f^*(s)ds \right\|_{L_q(0,r)} + \sup_{r>0} r^{-\frac{\lambda}{q}} \left\| t^{\frac{1}{p}-\frac{1}{q}} \int_t^\infty \frac{f^*(s)}{s} ds \right\| \right.$$
$$= I_1 + I_2.$$

I_1 can be estimated using the same method as in the proof of the boundedness of the maximal operator on $M^{loc}_{p,q;\lambda}$.

□

Proof.

Let us estimate I_2 :

$$I_2 = C \sup_{r>0} r^{-\frac{\lambda}{q}} \left\| t^{\frac{1}{p}-\frac{1}{q}} \int_t^\infty \frac{f^*(s)}{s} ds \right\|_{L_q(0,r)} = C \|\mathcal{A}_{(\frac{1}{p}-\frac{1}{q})} g\|_{LM_{q,\lambda}(0,\infty)},$$

where $g(t) = t^{\frac{1}{p}-\frac{1}{q}} f^*(t)$. Since $\frac{1}{p} - \frac{\lambda}{q} > 0$, for $\beta = \frac{1}{p} - \frac{1}{q}$ the inequality $\beta > \frac{\lambda}{q} - \frac{1}{q}$ holds. By Theorem C we get

$$\|\mathcal{A}_{(\frac{1}{p}-\frac{1}{q})} g\|_{LM_{q,\lambda}(0,\infty)} \leq C \|g\|_{LM_{q,\lambda}(0,\infty)}$$

$$= C \sup_{r>0} r^{-\frac{\lambda}{q}} \|t^{\frac{1}{p}-\frac{1}{q}} f^*(t)\|_{L_q(0,r)} = C \|f\|_{M^{loc}_{p,q;\lambda}}.$$

Therefore we get $I_2 \leq C\|f\|_{M^{loc}_{p,q;\lambda}}$. Consequently we obtain the boundedness of T in $M^{loc}_{p,q;\lambda}$. □

Proof.

(ii) For the limiting case $p = \frac{q}{q+\lambda}$, $1 < q < \infty$, suppose $f \in M^{loc}_{p,q;\lambda}$. From the definition of norm in weak local Morrey-Lorentz spaces and by using the inequality (10) and Minkowski's inequality we get

$$\begin{aligned}
\|Tf\|_{WM^{loc}_{\frac{q}{q+\lambda},q;\lambda}} &= \sup_{r>0} r^{-\frac{\lambda}{q}} \|t^{\frac{q+\lambda}{q}-\frac{1}{q}} (Tf)^*(t)\|_{WL_q(0,r)} \\
&\leq C \sup_{r>0} r^{-\frac{\lambda}{q}} \left\| t^{\frac{\lambda-1}{q}} \int_0^t f^*(s)ds \right\|_{WL_q(0,r)} \\
&+ C \sup_{r>0} r^{-\frac{\lambda}{q}} \left\| t^{1+\frac{\lambda-1}{q}} \int_t^\infty \frac{f^*(s)}{s} ds \right\|_{WL_q(0,r)} = N_1 + N_2.
\end{aligned}$$

N_1 can be estimated using the same method as in the proof of the weak boundedness of the maximal operator on $M^{loc}_{p,q;\lambda}$. □

Proof.

Let us estimate N_2 :

$$N_2 = C\sup_{r>0} r^{-\frac{\lambda}{q}} \left\| t^{1+\frac{\lambda-1}{q}} \int_t^\infty \frac{f^*(s)}{s} ds \right\|_{WL_q(0,r)}$$
$$= C\|\mathcal{A}_\beta h\|_{WLM_{q,\lambda}(0,\infty)},$$

where $\beta = 1 + \frac{\lambda-1}{q}$ and $h(t) = t^{1+\frac{\lambda-1}{q}} f^*(t)$. From Lemma 2 the operator $\mathcal{A}_\beta$ is bounded from the Morrey spaces $LM_{q,\lambda}(0,\infty)$ to $WLM_{q,\lambda}(0,\infty)$. Then,

$$N_2 \le C\|\mathcal{A}_\beta h\|_{WLM_{q,\lambda}(0,\infty)} \le C\|h\|_{LM_{q,\lambda}(0,\infty)}$$
$$= C\sup_{r>0} r^{-\frac{\lambda}{q}} \|t^{1+\frac{\lambda-1}{q}} f^*(t)\|_{L_q(0,r)} = C\|f\|_{M^{loc}_{\frac{q}{q+\lambda},q;\lambda}}.$$

Then we obtain the boundedness of the operator T from $M^{loc}_{\frac{q}{q+\lambda},q;\lambda}$ to $WM^{loc}_{\frac{q}{q+\lambda},q;\lambda}$. □

Proof.

(iii) For the limiting case $p = \frac{q}{\lambda}$, $1 \le q \le \infty$ and $0 \le \lambda < 1$, suppose $f \in M^{loc}_{p,q;\lambda}$.

Since the operator T^0 is bounded from L_∞ to BMO, then the inequality

$$\|T^0 f\|_{BMO} \le C\|f\|_{M^{loc}_{\frac{q}{\lambda},q;\lambda}} = C\|f\|_{L_\infty}$$

holds (Dyn'kin, 1991) which proves that Calderon-Zygmund operator T^0 is bounded from $M^{loc}_{\frac{q}{\lambda},q;\lambda}$ to BMO. □

Theorem

Suppose that $f \in M^{loc}_{p,q;\lambda}$, $1 \le q \le \infty$, $0 \le \lambda < 1$, $\frac{q}{q+\lambda} \le p \le \frac{q}{\lambda}$ *and the inequality* (2) *holds, then the maximal Calderon-Zygmund integral* $\mathcal{T} f(x)$ *is finite almost everywhere. Furthermore,*
(*i*) *If* $1 \le q < \infty$, $\frac{q}{q+\lambda} < p < \frac{q}{\lambda}$, *then the operator* $\mathcal{T}$ *is bounded in the local Morrey-Lorentz space* $M^{loc}_{p,q;\lambda}$.
(*ii*) *If* $1 < q < \infty$, $p = \frac{q}{q+\lambda}$, *then the operator* $\mathcal{T}$ *is bounded from* $M^{loc}_{p,q;\lambda}$ *to the weak local Morrey- Lorentz space* $WM^{loc}_{p,q;\lambda}$.
(*iii*) *If* $1 \le q \le \infty$, $p = \frac{q}{\lambda}$, *then the operator* $\mathcal{T}$ *is bounded from* $M^{loc}_{p,q;\lambda}$ *to BMO.*

Proof.

Let $1 \leq q \leq \infty$, $0 \leq \lambda < 1$ and $\frac{q}{q+\lambda} \leq p \leq \frac{q}{\lambda}$. Since f satisfies (2), by Theorem C the maximal Calderon-Zygmund operator $\mathcal{T}f(x)$ is finite almost everywhere.

The proof of the statements (i) and (ii) of this theorem can be easily obtained from the inequality (9) and using the same method in the proof the boundedness of the Calderon-Zygmund operator T.

□

Proof.

(iii) For the limiting case $1 \le q \le \infty$, $p = \frac{q}{\lambda}$ and $0 \le \lambda < 1$, suppose $f \in M^{loc}_{p,q;\lambda}$. Since the operator $\mathcal{T}$ is bounded from L_∞ to BMO and $M^{loc}_{\frac{q}{\lambda},q;\lambda} \equiv L_\infty$, the inequality

$$\|\mathcal{T} f\|_{BMO} \le C\|f\|_{M^{loc}_{\frac{q}{\lambda},q;\lambda}}$$

holds (see Bennett, De Vore and Sharpley, 1980) which proves that the operator $\mathcal{T}$ is bounded from $M^{loc}_{\frac{q}{\lambda},q;\lambda}$ to BMO.

Thus the proof of the theorem is completed. □

Riesz Potential

In this presentation we talk about the boundedness of the Riesz potential I_α defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \quad 0 < \alpha < n, \, f \in L_1^{loc}(\mathbb{R}^n),$$

in the local Morrey-Lorentz spaces $M_{p,q;\lambda}^{loc}(\mathbb{R}^n)$. We will find the necessary and sufficient conditions for the boundedness of I_α in the spaces $M_{p,q;\lambda}^{loc}(\mathbb{R}^n)$. We will apply these results to the boundedness of particular operators such as the fractional maximal operator, fractional Marcinkiewicz operator and fractional powers of some analytic semigroups on $M_{p,q;\lambda}^{loc}(\mathbb{R}^n)$.

Riesz Potential

For each measurable function f on $(0,\infty)$ and each $t>0$, the following operator

$$(S_\alpha f)(t) = t^{\frac{\alpha}{n}-1}\int_0^t f(s)ds + \int_t^\infty s^{\frac{\alpha}{n}-1} f(s)ds$$

was defined by A.P. Calderon (Calderon 1966).

Theorem

Theorem H. *(Oneil 1963, Sawyer 1990) If the condition*

$$(S_\alpha f^*)(1) = \int_0^1 f^*(s)ds + \int_1^\infty s^{\frac{\alpha}{n}-1} f^*(s)ds < \infty, \tag{11}$$

holds for $f \in L_1^{loc}(\mathbb{R}^n)$, *then the Riesz potential* $(I_\alpha f)(x)$, $x \in \mathbb{R}^n$ *exists almost everywhere. Furthermore, the inequality*

$$(I_\alpha f)^*(t) \le C S_\alpha(f^*)(t),\ 0 < t < \infty \tag{12}$$

is valid, where C *is a constant independent of* f *and* t.

We will need the following two Hardy operators to prove our main theorem.

Definition

Let φ be a measurable function on $(0,\infty)$ and γ and β be real numbers. The weighted Hardy operators with power weights acting on φ are defined by

$$H^{\beta}_{(\gamma)}\varphi(t) = t^{\gamma+\beta-1}\int_0^t \frac{\varphi(y)}{y^{\gamma}}dy \text{ and } \mathcal{H}^{\beta}_{(\gamma)}\varphi(t) = t^{\gamma+\beta}\int_t^{\infty} \frac{\varphi(y)}{y^{\gamma+1}}dy.$$

In the following two lemmas we give the boundedness of the Hardy operators $H^{\alpha}_{(\gamma)}$ and $\mathcal{H}^{\alpha}_{(\gamma)}$ on Morrey and weak Morrey spaces.

Lemma

(N. Samko 2009) Let $0 < \lambda < 1$, $0 < \beta < 1 - \lambda$, $1 \le r < \frac{1-\lambda}{\beta}$ *and* $\frac{1}{r} - \frac{1}{s} = \frac{\beta}{1-\lambda}$. *If* $\gamma < \frac{1}{r'} + \frac{\lambda}{r}$, *then* $H^{\beta}_{(\gamma)}$ *is bounded from* $LM_{r,\lambda}(0,\infty)$ *to* $LM_{s,\lambda}(0,\infty)$, *and if* $\gamma > \frac{\lambda-1}{r}$, *then the operator* $\mathcal{H}^{\beta}_{(\gamma)}$ *is bounded from* $LM_{r,\lambda}(0,\infty)$ *to* $LM_{s,\lambda}(0,\infty)$.

Lemma

(Guliyev, Kucukaslan, Aykol, Serbetci, 2017) Let $0 < \lambda < 1$, $0 < \beta < 1 - \lambda$, $1 \le r < \frac{1-\lambda}{\beta}$ *and* $\frac{1}{r} - \frac{1}{s} = \frac{\beta}{1-\lambda}$. *If* $\gamma = \frac{1}{r'} + \frac{\lambda}{r}$, *then* $H^{\beta}_{(\gamma)}$ *is bounded from* $LM_{r,\lambda}(0,\infty)$ *to* $WLM_{s,\lambda}(0,\infty)$, *and if* $\gamma = \frac{\lambda-1}{r}$, *then the operator* $\mathcal{H}^{\beta}_{(\gamma)}$ *is bounded from* $LM_{r,\lambda}(0,\infty)$ *to* $WLM_{s,\lambda}(0,\infty)$.

The following is a well known theorem about Riesz potential.

Theorem

Theorem J. *Let* $0 < \alpha < n$, $1 \leq p \leq \frac{n}{\alpha}$, $p < q < \infty$, $1 \leq r \leq \infty$ *and* $f \in L_{p,r}(\mathbb{R}^n)$ *satisfies the condition* (11). *Then the Riesz potential* $I_\alpha f$ *exists almost everywhere. Furthermore,*
(i) If $1 < p < \frac{n}{\alpha}$, $1 \leq r \leq s \leq \infty$, *then the condition* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$ *is necessary and sufficient for the boundedness of the operator* I_α *from the Lorentz spaces* $L_{p,r}$ *to* $L_{q,s}$.
(ii) If $p = 1$, *then the condition* $1 - \frac{1}{q} = \frac{\alpha}{n}$ *is necessary and sufficient for the boundedness of the operator* I_α *from the Lorentz spaces* $L_{1,r}$ *to* WL_q.

Riesz Potential

Theorem

Let $0 < \alpha < n$, $0 \leq \lambda < 1$, $1 \leq r \leq s \leq \infty$, $1 \leq q \leq \infty$, $\frac{r}{r+\lambda} \leq p \leq \left(\frac{\lambda}{r} + \frac{\alpha}{n}\right)^{-1}$ *and* $f \in M^{loc}_{p,r;\lambda}(\mathbb{R}^n)$ *satisfies the condition* (11). *Then the Riesz potential* $I_\alpha f$ *exists almost everywhere. Furthermore,*

(*i*) *If* $\frac{r}{r+\lambda} < p < \left(\frac{\lambda}{r} + \frac{\alpha}{n}\right)^{-1}$, *then the condition* $\frac{1}{p} - \frac{1}{q} = \lambda\left(\frac{1}{r} - \frac{1}{s}\right) + \frac{\alpha}{n}$ *is necessary and sufficient for the boundedness of the operator* I_α *from the spaces* $M^{loc}_{p,r;\lambda}$ *to* $M^{loc}_{q,s;\lambda}$.

(*ii*) *If* $p = \frac{r}{r+\lambda}$, *then the condition* $1 - \frac{1}{q} = \frac{\alpha}{n} - \frac{\lambda}{s}$ *is necessary and sufficient for the boundedness of the operator* I_α *from the spaces* $M^{loc}_{p,r;\lambda}$ *to* $WM^{loc}_{q,s;\lambda}$.

Proof of the theorem

If f satisfies (11), then from Theorem H the Riesz potential $I_\alpha f(x)$, $x \in \mathbb{R}^n$ exists almost everywhere.
(i)

Sufficiency.

Let $\frac{r}{r+\lambda} < p < \left(\frac{\lambda}{r} + \frac{\alpha}{n}\right)^{-1}$. From the inequality (12) we get

$$\|I_\alpha f\|_{M^{loc}_{q,s,\lambda}} = \sup_{t>0} t^{-\frac{\lambda}{s}} \|\tau^{\frac{1}{q}-\frac{1}{s}} (I_\alpha f)^*(\tau)\|_{L_s(0,t)}$$

$$\leq C \sup_{t>0} t^{-\frac{\lambda}{s}} \|\tau^{\frac{1}{q}-\frac{1}{s}} \left(\tau^{\frac{\alpha}{n}-1} \int_0^\tau f^*(y)dy + \int_\tau^\infty y^{\frac{\alpha}{n}-1} f^*(y)dy\right) \|_{L_s(0,t)}$$

$$\leq C \sup_{t>0} t^{-\frac{\lambda}{s}} \|\tau^{\frac{1}{q}-\frac{1}{s}+\frac{\alpha}{n}-1} \int_0^\tau f^*(y)dy\|_{L_s(0,t)}$$

$$+ C \sup_{t>0} t^{-\frac{\lambda}{s}} \|\tau^{\frac{1}{q}-\frac{1}{s}} \int_\tau^\infty y^{\frac{\alpha}{n}-1} f^*(y)dy\|_{L_s(0,t)} = I_1 + I_2.$$

Proof of the theorem

We have

$$I_1 = C \sup_{t>0} t^{-\frac{\lambda}{s}} \|\tau^{\frac{1}{q}-\frac{1}{s}+\frac{\alpha}{n}-1} \int_0^\tau f^*(y)dy\|_{L_s(0,t)} = C\|H^\beta_{(\gamma)} g\|_{LM_{s,\lambda}(0,\infty)}.$$

Consider the Hardy operator $H^\beta_{(\gamma)}$ such that $\gamma = \frac{1}{p} - \frac{1}{r}$ and $g(t) = t^{\frac{1}{p}-\frac{1}{r}} f^*(t)$. Since the operator $H^\beta_{(\gamma)}$ is bounded from the Morrey space $LM_{r,\lambda}(0,\infty)$ to $LM_{s,\lambda}(0,\infty)$, we get

$$\begin{aligned} I_1 &= C\|H^\beta_{(\gamma)} g\|_{LM_{s,\lambda}(0,\infty)} \leq C\|g\|_{LM_{r,\lambda}(0,\infty)} \\ &= C \sup_{t>0} t^{-\frac{\lambda}{r}} \|\tau^{\frac{1}{p}-\frac{1}{r}} f^*(\tau)\|_{L_r(0,t)} = C\|f\|_{M^{loc}_{p,r;\lambda}}. \end{aligned} \tag{13}$$

Proof of the theorem

$$I_2 = C \sup_{t>0} t^{-\frac{\lambda}{s}} \|\tau^{\frac{1}{q}-\frac{1}{s}} \int_\tau^\infty y^{\frac{\alpha}{n}-1} f^*(y)dy\|_{L_s(0,t)} = C\|\mathcal{H}^\beta_{(\gamma)} g\|_{LM_{s,\lambda}(0,\infty)}.$$

Consider the Hardy operator $H^\beta_{(\gamma)}$ such that $\gamma = \frac{1}{p} - \frac{1}{r} - \frac{\alpha}{n}$ and $g(t) = t^{\frac{1}{p}-\frac{1}{r}} f^*(t)$. Since the operator $H^\beta_{(\gamma)}$ is bounded from the Morrey space $LM_{r,\lambda}(0,\infty)$ to $LM_{s,\lambda}(0,\infty)$ we get

$$\begin{aligned} I_2 &= C\|\mathcal{H}^\beta_{(\gamma)} g\|_{LM_{s,\lambda}(0,\infty)} \leq C\|g\|_{LM_{r,\lambda}(0,\infty)} \\ &= C \sup_{t>0} t^{-\frac{\lambda}{r}} \|\tau^{\frac{1}{p}-\frac{1}{r}} f^*(\tau)\|_{L_r(0,t)} = C\|f\|_{M^{loc}_{p,r;\lambda}}. \end{aligned} \tag{14}$$

From the inequalities (13) and (14) we obtain the boundedness of the operator I_α from $M^{loc}_{p,r;\lambda}$ to $M^{loc}_{q,s;\lambda}$.

Necessity.

Suppose that the operator I_α is bounded from $M^{loc}_{p,r;\lambda}$ to $M^{loc}_{q,s;\lambda}$, and $\frac{r}{r+\lambda} \le p \le \left(\frac{\lambda}{r} + \frac{\alpha}{n}\right)^{-1}$.
Define $f_\tau(x) =: f(\tau x)$ for $\tau > 0$. Then $f^*_\tau(t) = f^*(t\tau^n)$ and

$$\|f_\tau\|_{M^{loc}_{p,r;\lambda}} = \tau^{-n\left(\frac{1}{p}-\frac{\lambda}{r}\right)}\|f\|_{M^{loc}_{p,r;\lambda}}.$$

Also,

$$(I_\alpha f_\tau)(x) = \tau^{-\alpha}(I_\alpha f)(\tau^n x),$$
$$(I_\alpha f_\tau)^*(t) = \tau^{-\alpha}(I_\alpha f)^*(t\tau^n)$$

and

$$\|I_\alpha f_\tau\|_{M^{loc}_{q,s;\lambda}} = \tau^{-\alpha-n(\frac{1}{q}-\frac{\lambda}{s})}\|I_\alpha f\|_{M^{loc}_{q,s;\lambda}}.$$

Therefore we get $\frac{1}{p} - \frac{1}{q} = \lambda\left(\frac{1}{r} - \frac{1}{s}\right) + \frac{\alpha}{n}$.

(ii)

Sufficiency.

For the limiting case $p = \frac{r}{r+\lambda}$, $1 \le r \le s < \infty$, suppose $f \in M^{loc}_{p,r;\lambda}$. By using the inequality (12) and Minkowski's inequality we get

$$\begin{aligned}
\|I_\alpha f\|_{WM^{loc}_{q,s;\lambda}} &= \sup_{t>0} t^{-\frac{\lambda}{s}} \|\tau^{\frac{1}{q}-\frac{1}{s}} (I_\alpha f)^*(\tau)\|_{WL_s(0,t)} \\
&\le C \sup_{t>0} t^{-\frac{\lambda}{s}} \|\tau^{\frac{1}{q}-\frac{1}{s}} \Big(\tau^{\frac{\alpha}{n}-1} \int_0^\tau f^*(y)dy + \int_\tau^\infty f^*(y) y^{\frac{\alpha}{n}-1} dy \Big) \\
&\le C \sup_{t>0} t^{-\frac{\lambda}{s}} \|\tau^{\frac{1}{q}-\frac{1}{s}+\frac{\alpha}{n}-1} \int_0^\tau f^*(y)dy\|_{WL_s(0,t)} \\
&\quad + C \sup_{t>0} t^{-\frac{\lambda}{s}} \|\tau^{\frac{1}{q}-\frac{1}{s}} \int_\tau^\infty f^*(y) y^{\frac{\alpha}{n}-1} dy\|_{WL_s(0,t)} \\
&= N_1 + N_2.
\end{aligned}$$

Proof of the theorem

We have

$$N_1 = C \sup_{t>0} t^{-\frac{\lambda}{s}} \|\tau^{\frac{1}{q}-\frac{1}{s}+\frac{\alpha}{n}-1} \int_0^\tau f^*(y)dy\|_{WL_s(0,t)}$$
$$= C\|H^\beta_{(\gamma)} h\|_{WLM_{s,\lambda}(0,\infty)}.$$

Consider the Hardy operator $H^\beta_{(\gamma)}$ such that $\gamma = 1 + \frac{\lambda-1}{r}$ and $h(t) = t^{1+\frac{\lambda-1}{r}} f^*(t)$. From the boundedness of the operator $H^\beta_{(\gamma)}$ we have

$$N_1 = C\|H^\beta_{(\gamma)} h\|_{WLM_{s,\lambda}(0,\infty)} \leq C\|h\|_{LM_{r,\lambda}(0,\infty)}$$
$$= C \sup_{t>0} t^{-\frac{\lambda}{r}} \|\tau^{1+\frac{\lambda-1}{r}} f^*(\tau)\|_{L_r(0,t)} = C\|f\|_{M^{loc}_{p,r;\lambda}}. \qquad (15)$$

Proof of the theorem

Now we consider N_2.

$$N_2 = C \sup_{t>0} t^{-\frac{\lambda}{s}} \|\tau^{\frac{1}{q}-\frac{1}{s}} \int_\tau^\infty f^*(y) y^{\frac{\alpha}{n}-1} dy\|_{WL_s(0,t)} = C\|\mathcal{H}^\beta_{(\gamma)} h\|_{WL_{q,\lambda}(0,\infty)}.$$

Consider the Hardy operator $\mathcal{H}^\beta_{(\gamma)}$ such that $\gamma = 1 + \frac{\lambda-1}{r} - \frac{\alpha}{n}$ and $h(t) = t^{1+\frac{\lambda-1}{r}} f^*(t)$.

Since the operator $\mathcal{H}^\beta_{(\gamma)}$ is bounded from the Morrey spaces $LM_{r,\lambda}(0,\infty)$ to $WLM_{s,\lambda}(0,\infty)$ we get

$$\begin{aligned} N_2 &= C\|\mathcal{H}^\beta_{(\gamma)} h\|_{WLM_{s,\lambda}(0,\infty)} \le C\|h\|_{LM_{r,\lambda}(0,\infty)} \\ &= C \sup_{t>0} t^{-\frac{\lambda}{r}} \|\tau^{1+\frac{\lambda-1}{r}} f^*(\tau)\|_{L_r(0,t)} = C\|f\|_{M^{loc}_{p,r;\lambda}}. \end{aligned} \tag{16}$$

From the inequalities (15) and (16) we obtain the boundedness of the Riesz potential operator I_α from $M^{loc}_{p,r;\lambda}$ to $WM^{loc}_{q,s;\lambda}$.

Proof of the theorem

Necessity.

Suppose that the operator I_α is bounded from $M^{loc}_{p,r;\lambda}$ to $WM^{loc}_{q,s;\lambda}$ for $p = \frac{r}{r+\lambda}$. Define $f_\tau(x) =: f(\tau x)$ for $\tau > 0$.

$$\|f_\tau\|_{M^{loc}_{\frac{r}{r+\lambda},r;\lambda}} = \tau^{-n}\|f\|_{M^{loc}_{\frac{r}{r+\lambda},r;\lambda}}$$

and

$$\|I_\alpha f_\tau\|_{WM^{loc}_{q,s;\lambda}} = \tau^{-\alpha-n(\frac{1}{q}-\frac{\lambda}{s})}\|I_\alpha f\|_{WM^{loc}_{q,s;\lambda}}.$$

Therefore we get the equality $1 - \frac{1}{q} = \frac{\alpha}{n} - \frac{\lambda}{s}$ and the proof of the theorem is completed.

Maximal Bochner-Riesz operator

Let $\delta > (n-1)/2$, $B_r^\delta(f)\hat{}(\xi) = (1 - r^2|\xi|^2)_+^\delta \hat{f}(\xi)$ and $B_r^\delta(x) = r^{-n}B^\delta(x/r)$ for $r > 0$. The maximal Bochner-Riesz operator is defined by (see L.Z. Liu and S.Z. Lu (2003) and Y. Liu and D. Chen (2008))

$$B_{\delta,*}(f)(x) = \sup_{r>0} |B_r^\delta(f)(x)|.$$

Let H be the space $H = \{h : \|h\| = \sup_{r>0} |h(r)| < \infty\}$, then it is clear that $B_{\delta,*}(f)(x) = \|B_r^\delta(f)(x)\|$. By the condition on B_r^δ (J. Garcia-Cuerva and J.L. Rubio de Francia (1985)), we have

$$|B_r^\delta(x-y)| \leq Cr^{-n}(1 + |x-y|/r)^{-(\delta+(n+1)/2)} \leq r^{-n},$$

and

$$B_{\delta,*}(f)(x) \leq C \sup_{r>0} \frac{1}{r^n} \int_{B(x,r)} |f(y)|dy = CMf(x).$$

Since the maximal operator M is bounded on the spaces $M^{loc}_{p,q;\lambda}$, we get the following corollary.

Corollary

Let $1 \leq q \leq \infty$, $0 \leq \lambda < 1$ and $\frac{q}{q+\lambda} \leq p \leq \infty$.
(i) If $\frac{q}{q+\lambda} < p < \infty$, then the Bochner-Riesz operator B^{δ}_{r} is bounded on the local Morrey-Lorentz space $M^{loc}_{p,q;\lambda}$.
(ii) If $p = \frac{q}{q+\lambda}$, then the Bochner-Riesz operator B^{δ}_{r} is bounded from $M^{loc}_{p,q;\lambda}$ to the weak local Morrey- Lorentz space $WM^{loc}_{p,q;\lambda}$.
(iii) If $p = q = \infty$, then the Bochner-Riesz operator B^{δ}_{r} is bounded on $L_{\infty}(\mathbb{R}^n)$.

In this section, we apply the our main theorem to the fractional maximal operator, fractional Marcinkiewicz operator and the fractional powers of some analytic semigroups.

Fractional maximal operator

For $0 \leq \alpha < n$, we define the fractional maximal operator

$$M_\alpha f(x) = \sup_{t>0} |B(x,t)|^{\frac{\alpha}{n}-1} \int_{B(x,t)} |f(y)| dy,$$

where $B(x,t)$ is the open ball centered at x of radius t for $x \in \mathbb{R}^n$ and $|B(x,t)|$ is the Lebesgue measure of $B(x,t)$ such that $|B(x,t)| = \omega_n t^n$ in which ω_n denotes the volume of unit ball in $\mathbb{R}^n$.

Some applications

It is well known that the following inequality holds:

$$M_\alpha f(x) \leq \omega_n^{\frac{\alpha}{n}-1}(I_\alpha |f|)(x).$$

From this inequality we get the following corollary.

Corollary

Let $0 < \alpha < n$, $0 \leq \lambda < 1$, $1 \leq r \leq s \leq \infty$, $1 \leq q \leq \infty$,
$\frac{r}{r+\lambda} \leq p \leq \left(\frac{\lambda}{r} + \frac{\alpha}{n}\right)^{-1}$ *and* $\frac{1}{p} - \frac{1}{q} = \lambda\left(\frac{1}{r} - \frac{1}{s}\right) + \frac{\alpha}{n}$.
(i) If $\frac{r}{r+\lambda} < p < \left(\frac{\lambda}{r} + \frac{\alpha}{n}\right)^{-1}$, *then the condition* $\frac{1}{p} - \frac{1}{q} = \lambda\left(\frac{1}{r} - \frac{1}{s}\right) + \frac{\alpha}{n}$ *is necessary and sufficient for the boundedness of the fractional maximal operator* M_α *from the space* $M^{loc}_{p,r;\lambda}$ *to* $M^{loc}_{q,s;\lambda}$.
(ii) If $p = \frac{r}{r+\lambda}$, *then the condition* $1 - \frac{1}{q} = \frac{\alpha}{n} - \frac{\lambda}{s}$ *necessary and sufficient for the boundedness of the operator* M_α *from the space* $M^{loc}_{p,r;\lambda}$ *to* $WM^{loc}_{q,s;\lambda}$.

Fractional Marcinkiewicz operator

Let $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ be the unit sphere in $\mathbb{R}^n$ equipped with the Lebesgue measure $d\sigma$. Suppose that Ω satisfies the following conditions.

(a) Ω is the homogeneous function of degree zero on $\mathbb{R}^n \setminus \{0\}$, that is,

$$\Omega(tx) = \Omega(x), \text{ for any } t > 0, \, x \in \mathbb{R}^n \setminus \{0\}.$$

(b) Ω has mean zero on S^{n-1}, that is,

$$\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0.$$

(c) $\Omega \in \mathrm{Lip}_\gamma(S^{n-1})$, $0 < \gamma \leq 1$, that is there exists a constant $C > 0$ such that,

$$|\Omega(x') - \Omega(y')| \leq C|x' - y'|^\gamma \text{ for any } x', y' \in S^{n-1}.$$

In 1958, Stein defined the Marcinkiewicz integral of higher dimension μ_Ω as

$$\mu_{\Omega,\alpha}(f)(x) = \left(\int_0^\infty |F_{\Omega,\alpha,t}(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F_{\Omega,\alpha,t}(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1-\alpha}} f(y)dy.$$

By Minkowski inequality and the conditions on Ω, we get

$$\mu_{\Omega,\alpha}(f)(x) \le I_\alpha(|f|)(x).$$

Then we have the following corollary.

Corollary

Let $0 < \alpha < n$, $0 \leq \lambda < 1$, $1 \leq r \leq s \leq \infty$, $1 \leq q \leq \infty$ *,*
$\frac{r}{r+\lambda} \leq p \leq \left(\frac{\lambda}{r} + \frac{\alpha}{n}\right)^{-1}$ *and* $\frac{1}{p} - \frac{1}{q} = \lambda\left(\frac{1}{r} - \frac{1}{s}\right) + \frac{\alpha}{n}$.
(i) *If* $\frac{r}{r+\lambda} < p < \left(\frac{\lambda}{r} + \frac{\alpha}{n}\right)^{-1}$ *and* $\frac{1}{p} - \frac{1}{q} = \lambda\left(\frac{1}{r} - \frac{1}{s}\right) + \frac{\alpha}{n}$*, then the Marcinkiewicz operator* μ_Ω *is bounded from the space* $M^{loc}_{p,r;\lambda}$ *to* $M^{loc}_{q,s;\lambda}$.
(ii) *If* $p = \frac{r}{r+\lambda}$ *and* $1 - \frac{1}{q} = \frac{\alpha}{n} - \frac{\lambda}{s}$*, then the operator* μ_Ω *is bounded from the space* $M^{loc}_{p,r;\lambda}$ *to* $WM^{loc}_{q,s;\lambda}$.

Fractional powers of some analytic semigroups

Suppose that L is a linear operator on L_2 which generates and analytic semigoup e^{-tL} with the kernel $p_t(x, y)$ satisfying a Gaussian upper bound, that is,

$$|p_t(x, y)| \leq \frac{c_1}{t^{n/2}} e^{-c_2 \frac{|x-y|^2}{t}} \tag{17}$$

for $x, y \in \mathbb{R}^n$ and all $t > 0$, where $c_1, c_2 > 0$ are independent of x, y and t.
For $0 < \alpha < n$, the fractional powers $L^{-\alpha/2}$ of the operator L are defined by

$$L^{-\alpha/2} f(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty e^{-tL} f(x) \frac{dt}{t^{-\alpha/2+1}}.$$

Some applications

Note that if $L = -\Delta$ is the Laplacian on $\mathbb{R}^n$, then $L^{-\alpha/2}$ is the Riesz potential I_α. Property (17) is satisfied for large classes of differential operators. Since the semigroup e^{-tL} has kernel $p_t(x, y)$ which satisfies condition (17), it follows that

$$|L^{-\alpha/2} f(x)| \leq C I_\alpha(|f|)(x).$$

Hence we get the following corollary.

Corollary

Let $0 < \alpha < n$, $0 \leq \lambda < 1$, $1 \leq r \leq s \leq \infty$, $1 \leq q \leq \infty$ $\frac{r}{r+\lambda} \leq p \leq \left(\frac{\lambda}{r} + \frac{\alpha}{n}\right)^{-1}$ and $\frac{1}{p} - \frac{1}{q} = \lambda\left(\frac{1}{r} - \frac{1}{s}\right) + \frac{\alpha}{n}$.

(i) If $\frac{r}{r+\lambda} < p < \left(\frac{\lambda}{r} + \frac{\alpha}{n}\right)^{-1}$ and $\frac{1}{p} - \frac{1}{q} = \lambda\left(\frac{1}{r} - \frac{1}{s}\right) + \frac{\alpha}{n}$, then the operator $L^{-\alpha/2}$ is bounded from the space $M^{loc}_{p,r;\lambda}$ to $M^{loc}_{q,s;\lambda}$.

(ii) If $p = \frac{r}{r+\lambda}$ and $1 - \frac{1}{q} = \frac{\alpha}{n} - \frac{\lambda}{s}$, then the operator $L^{-\alpha/2}$ is bounded from the space $M^{loc}_{p,r;\lambda}$ to $WM^{loc}_{q,s;\lambda}$.

- D.R. Adams, *A note on Riesz potentials*, Duke Math. J., **42** (1975), 765–778.
- D.R. Adams and L.I. Hedberg, *Function spaces and potential theory*, Grundlehren der Mathematischen Wissenschaften 314. Springer-Verlag, Berlin, 1996.
- D.R. Adams and J.L. Lewis, *On Morrey-Besov inequalities.* Studia Math. **74** (1982), 169–182.
- K.F. Andersen and B. Muckenhoupt, *Weighted weak type Hardy inequalities with applications to Hilbert transforms and maximal functions*, Studia Math., **72** (1982), 9–26.
- C. Aykol, V.S. Guliyev and A. Serbetci, *Boundedness of the maximal operator in the local Morrey-Lorentz spaces,* Jour. Inequal. Appl., 2013, 2013:346.
- C. Aykol, V.S. Guliyev, A. Kucukaslan and A. Serbetci, *The boundedness of Hilbert transform in the local Morrey-Lorentz*

spaces, Integral Transforms Spec. Funct. **27** (2016), no. 4, 318–330.

C. Bennett and R. Sharpley, *Interpolation of Operators*, Academic Press, Boston, 1988.

V.I. Burenkov, H.V. Guliyev, V.S. Guliyev, *Necessary and sufficient conditions for boundedness of the fractional maximal operators in the local Morrey-type spaces*, J. Comput. Appl. Math., **208** (2007), no. 1, 280–301.

V.I. Burenkov and V.S. Guliyev, *Necessary and sufficient conditions for the boundedness of the Riesz operator in local Morrey-type spaces*, Potential Anal., **30** (2009), no. 3, 211–249.

A.P. Calderón, *Spaces between L^1 and L^∞ and the theorem of Marcinkiewicz*, Studia Math. **26** (1966), 273–299.

V.S. Guliyev, *Integral operators on function spaces on the homogeneous groups and on domains in $\mathbb{R}^n$*. Doctoral degree dissertation, Mat. Inst. Steklov, Moscow, 1994, 329 pp. (in Russian)

V.S. Guliyev, *Function spaces, Integral Operators and Two Weighted Inequalities on Homogeneous Groups. Some Applications*, Cashioglu, Baku, 1999, 332 pp. (in Russian)

V.S. Guliyev, A. Serbetci, I. Ekincioglu, *Necessary and sufficient conditions for the boundedness of rough B-fractional integral operators in the Lorentz spaces*. J. Math. Anal. Appl., **336** (2007), 425–437.

V.S. Guliyev, C. Aykol, A. Kucukaslan and A. Serbetci, *Maximal operator and Calderón-Zygmund operators in local Morrey-Lorentz spaces*, Integral Transforms Spec. Funct. 27 (2016), no. 11, 866–877.

V.S. Guliyev, A. Kucukaslan, C. Aykol and A. Serbetci, *Riesz potential in the local Morrey-Lorentz spaces and some applications*, Georgian Math. J., 2017 (accepted).

- K.-P. Ho, *Sobolev-Jawerth embedding of Triebel-Lizorkin-Morrey-Lorentz spaces and fractional integral operator on Hardy type spaces*, Math. Nachr. **287** (2014), no. 14-15, 1674–1686.
- S. Lu, Y. Ding and D. Yan, *Singular integrals and related topics*, World Scientific Publishing, Hackensack, NJ, USA, 2007.
- G. Mingione, *Gradient estimates below the duality exponent.* Math. Ann. **346** (2010), 571–627.
- G.G. Lorentz, *Some new function spaces*, Annals of Mathematics **51** (1950), 37–55.
- G.G. Lorentz, *On the theory of spaces* Λ. Pacific J. Math., **1** (1951), 411–429.
- M.A. Ragusa, *Embeddings for Morrey-Lorentz Spaces*, J. Optim. Theory Appl. **154** (2012), no. 2, 491–499.

N. Samko, *Weighted Hardy and singular operators in Morrey spaces*. J. Math. Anal. Appl., **350** (2009) 56–72.

N. Samko, *Weighted Hardy and potential operators in Morrey spaces*. J. Funct. Spaces Appl. 2012, Art. ID 678171, 21 pp.

E. Sawyer, *Boundedness of classical operators on classical Lorentz spaces.* Studia Math., **96** (1990), 145–158.

E.M. Stein, *On the functions of Littlewood-Paley, Lusin, and Marcinkiewicz*, Transactions of the American Mathematical Society. **88** (1958), 430–466.