On some Morrey regularity results for minimizers of variational integrals

Maria Alessandra Ragusa

Department of Mathematics and Computer Science,
University of Catania, Italy

Ahi Evran Universitesi, Kirsehir, July 11th, 2017

We treat the regularity problem for minimizers

$$u(x) : \Omega \subset R^m \to R^n$$

We treat the regularity problem for minimizers

$$u(x) : \Omega \subset R^m \to R^n$$

of quadratic growth functional

$$\int_\Omega A(x, u, Du)dx,$$

where $\Omega \subset \mathbb{R}^m$ is a bounded open set and

$$A : \Omega \times \mathbb{R}^n \times \mathbb{R}^{mn} \to \mathbb{R}.$$

We treat the regularity problem for minimizers

$$u(x) : \Omega \subset R^m \to R^n$$

of quadratic growth functional

$$\int_\Omega A(x, u, Du)dx,$$

where $\Omega \subset \mathbb{R}^m$ is a bounded open set and

$$A : \Omega \times \mathbb{R}^n \times \mathbb{R}^{mn} \to \mathbb{R}.$$

$A(\cdot, u, p)$ **is in the class VMO**,

We treat the regularity problem for minimizers

$$u(x) : \Omega \subset R^m \to R^n$$

of quadratic growth functional

$$\int_\Omega A(x, u, Du)dx,$$

where $\Omega \subset \mathbb{R}^m$ is a bounded open set and

$$A : \Omega \times \mathbb{R}^n \times \mathbb{R}^{mn} \to \mathbb{R}.$$

$A(\cdot, u, p)$ **is in the class VMO**, namely, is not assumed the continuity of $A(x, u, p)$ with respect to x.

We treat the regularity problem for minimizers

$$u(x) : \Omega \subset R^m \to R^n$$

of quadratic growth functional

$$\int_\Omega A(x, u, Du)dx,$$

where $\Omega \subset \mathbb{R}^m$ is a bounded open set and

$$A : \Omega \times \mathbb{R}^n \times \mathbb{R}^{mn} \to \mathbb{R}.$$

$A(\cdot, u, p)$ **is in the class VMO**, namely, is not assumed the continuity of $A(x, u, p)$ with respect to x.
Both **partial** and **global** regularity of the minimizer u are studied.

Definition

Let $1 < p < \infty, 0 < \lambda < m$.

Definition

Let $1 < p < \infty, 0 < \lambda < m$. A measurable function $f \in L^1_{\mathrm{loc}}(\Omega)$ is in the Morrey class $L^{p,\lambda}(\Omega)$ if the following norm is finite

Definition

Let $1 < p < \infty, 0 < \lambda < m$. A measurable function $f \in L^1_{\mathrm{loc}}(\Omega)$ is in the Morrey class $L^{p,\lambda}(\Omega)$ if the following norm is finite

$$\|f\|_{L^{p,\lambda}(\Omega)} = \sup_{\substack{x\in\Omega \\ 0<\rho<\mathrm{diam}\Omega}} \frac{1}{\rho^\lambda} \int_{\Omega\cap B(x,\rho)} |f(y)|^p dy,$$

Definition

Let $1 < p < \infty, 0 < \lambda < m$. A measurable function $f \in L^1_{\mathrm{loc}}(\Omega)$ is in the Morrey class $L^{p,\lambda}(\Omega)$ if the following norm is finite

$$\|f\|_{L^{p,\lambda}(\Omega)} = \sup_{\substack{x\in\Omega \\ 0<\rho<\mathrm{diam}\Omega}} \frac{1}{\rho^\lambda} \int_{\Omega\cap B(x,\rho)} |f(y)|^p dy,$$

where $B(x,\rho)$ ranges in the class of the balls centered in x with radius ρ.

Definition

Let $f \in L^1(\Omega)$ we set the integral mean $f_{x,R}$ by

$$f_{x,R} = \fint_{\Omega \cap B(x,R)} f(y)dy = \frac{1}{|\Omega \cap B(x,R)|} \int_{\Omega \cap B(x,R)} f(y)dy$$

Definition

Let $f \in L^1(\Omega)$ we set the integral mean $f_{x,R}$ by

$$f_{x,R} = \fint_{\Omega \cap B(x,R)} f(y)dy = \frac{1}{|\Omega \cap B(x,R)|} \int_{\Omega \cap B(x,R)} f(y)dy$$

where $|\Omega \cap B(x,R)|$ is the Lebesgue measure of $\Omega \cap B(x,R)$.

Definition

Let $f \in L^1(\Omega)$ we set the integral mean $f_{x,R}$ by

$$f_{x,R} = \fint_{\Omega \cap B(x,R)} f(y)dy = \frac{1}{|\Omega \cap B(x,R)|} \int_{\Omega \cap B(x,R)} f(y)dy$$

where $|\Omega \cap B(x,R)|$ is the Lebesgue measure of $\Omega \cap B(x,R)$.
If we are not interested in specifying which the center is, we only set f_R.

Let us now give the definition of bounded mean oscillation function (BMO) by John and Nirenberg

"On functions of bounded mean oscillation", C.P.A.M., 1961.

Let us now give the definition of bounded mean oscillation function (BMO) by John and Nirenberg
"On functions of bounded mean oscillation", C.P.A.M., 1961.

Definition

A function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^m)$ belongs to $BMO(\mathbb{R}^m)$ if the seminorm

Let us now give the definition of bounded mean oscillation function (BMO) by John and Nirenberg
"On functions of bounded mean oscillation", C.P.A.M., 1961.

Definition

A function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^m)$ belongs to $BMO(\mathbb{R}^m)$ if the seminorm

$$\|f\|_* \equiv \sup_{B(x,R)} \frac{1}{|B(x,R)|} \int_{B(x,R)} |f(y) - f_{x,R}| dy < \infty.$$

Let us now give the definition of bounded mean oscillation function (BMO) by John and Nirenberg
"On functions of bounded mean oscillation", C.P.A.M., 1961.

Definition

A function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^m)$ belongs to $BMO(\mathbb{R}^m)$ if the seminorm

$$\|f\|_* \equiv \sup_{B(x,R)} \frac{1}{|B(x,R)|} \int_{B(x,R)} |f(y) - f_{x,R}|dy < \infty.$$

Let us recall the definition of the space of Vanishing Mean Oscillation functions, by Sarason in

"On functions of vanishing mean oscillation", T.A.M.S., 1975.

Definition

Let $f \in BMO(\mathbb{R}^m)$ and

$$\eta(f,R) = \sup_{\rho \leq R} \frac{1}{|B_\rho|} \int_{B_\rho} |f(y) - f_\rho| dy$$

where $B\rho$ ranges over the class of the balls of $\mathbb{R}^m$ of radius ρ.

Definition

Let $f \in BMO(\mathbb{R}^m)$ and

$$\eta(f,R) = \sup_{\rho \leq R} \frac{1}{|B_\rho|} \int_{B_\rho} |f(y) - f_\rho| dy$$

where $B\rho$ ranges over the class of the balls of $\mathbb{R}^m$ of radius ρ.
A function $f \in VMO(\mathbb{R}^n)$ if

$$\lim_{R \to 0} \eta(f,R) = 0.$$

Let us make some remarks on *VMO* class.

Let us make some remarks on *VMO* class.
Defined by Sarason in 1975,

Let us make some remarks on *VMO* class.
Defined by Sarason in 1975, later considered by many others e.g. Chiarenza, Frasca and Longo

Let us make some remarks on *VMO* class.
Defined by Sarason in 1975, later considered by many others e.g. Chiarenza, Frasca and Longo that answer a question raised thirty years before by Miranda in

Let us make some remarks on *VMO* class.
Defined by Sarason in 1975, later considered by many others e.g. Chiarenza, Frasca and Longo that answer a question raised thirty years before by Miranda in
"Sulle equazioni ellittiche del secondo ordine a coefficienti discontinui", Ann. Mat. Pura Appl., 1963.

Miranda considers a linear elliptic equation where the coefficients $a_{ij} \in W^{1,m}(\Omega)$ and asks whether the gradient of the solution is bounded, if $p > m$.

Miranda considers a linear elliptic equation where the coefficients $a_{ij} \in W^{1,m}(\Omega)$ and asks whether the gradient of the solution is bounded, if $p > m$.
Chiarenza, Frasca and Longo suppose $a_{ij} \in VMO$

Miranda considers a linear elliptic equation where the coefficients $a_{ij} \in W^{1,m}(\Omega)$ and asks whether the gradient of the solution is bounded, if $p > m$.
Chiarenza, Frasca and Longo suppose $a_{ij} \in VMO$ and prove that Du is hölder continuous for all $p \in]1, +\infty[$.

Miranda considers a linear elliptic equation where the coefficients $a_{ij} \in W^{1,m}(\Omega)$ and asks whether the gradient of the solution is bounded, if $p > m$.
Chiarenza, Frasca and Longo suppose $a_{ij} \in VMO$ and prove that Du is hölder continuous for all $p \in]1, +\infty[$.

$$W^{1,m} \subset VMO,$$

Miranda considers a linear elliptic equation where the coefficients $a_{ij} \in W^{1,m}(\Omega)$ and asks whether the gradient of the solution is bounded, if $p > m$.
Chiarenza, Frasca and Longo suppose $a_{ij} \in VMO$ and prove that Du is hölder continuous for all $p \in]1, +\infty[$.

$$W^{1,m} \subset VMO,$$

Poincare's inequality:

Miranda considers a linear elliptic equation where the coefficients $a_{ij} \in W^{1,m}(\Omega)$ and asks whether the gradient of the solution is bounded, if $p > m$.
Chiarenza, Frasca and Longo suppose $a_{ij} \in VMO$ and prove that Du is hölder continuous for all $p \in]1, +\infty[$.

$$W^{1,m} \subset VMO,$$

Poincare's inequality:

$$\fint_B |f(x) - f_B| \leq c(m) \left(\int_B |\nabla u| dx \right)^{\frac{1}{m}}$$

and observing that the term on the right-hand side tends to zero as $|B| \to 0$.

Interior estimates obtained by Chiarenza, Frasca and Longo were extended to **boundary estimates.**

Interior estimates obtained by Chiarenza, Frasca and Longo were extended to **boundary estimates.** Later, many authors have used this space VMO to obtain regularity results for **P.D.E. and systems** with discontinuous coefficients.

Interior estimates obtained by Chiarenza, Frasca and Longo were extended to **boundary estimates.** Later, many authors have used this space VMO to obtain regularity results for **P.D.E. and systems** with discontinuous coefficients.
With this useful *VMO* assumption we investigate the regularity of the **minimizers for quadratic functionals.**

At first, let us talk about some **estimates in Morrey Spaces** for the derivatives of **"local minimizers"** of variational integrals of the form

$$\mathcal{A}(u,\Omega) = \int_{\Omega} F(x,u,Du)dx$$

where Ω is a domain in $\mathbb{R}^m$ and the integrand has the following form

$$F(x,u,Du) = A(x,u,\ g(x)\,h(u)\,Du\,Du).$$

At first, let us talk about some **estimates in Morrey Spaces** for the derivatives of **"local minimizers"** of variational integrals of the form

$$\mathcal{A}(u,\Omega) = \int_\Omega F(x,u,Du)dx$$

where Ω is a domain in $\mathbb{R}^m$ and the integrand has the following form

$$F(x,u,Du) = A(x,u,\, g(x)\, h(u)\, Du\, Du).$$

We are **not assuming the continuity** of A and g with respect to x.

A **“local minimizer”** of the functional $\mathcal{A}$ is a function $u \in W^{1,p}_{loc}(\Omega, \mathbb{R}^n)$ which satisfies

$$\mathcal{A}(u;\, supp\, \varphi) \leq \mathcal{A}(u + \varphi;\, supp\, \varphi)$$

for every $\varphi \in W^{1,p}_0(\Omega, \mathbb{R}^n)$.

A **“local minimizer”** of the functional $\mathcal{A}$ is a function $u \in W^{1,p}_{loc}(\Omega, \mathbb{R}^n)$ which satisfies

$$\mathcal{A}(u;\, supp\, \varphi) \leq \mathcal{A}(u+\varphi;\, supp\, \varphi)$$

for every $\varphi \in W^{1,p}_0(\Omega, \mathbb{R}^n)$.
Partial regularity for solutions of nonlinear elliptic systems has been studied by Morrey

A **"local minimizer"** of the functional $\mathcal{A}$ is a function $u \in W^{1,p}_{loc}(\Omega, \mathbb{R}^n)$ which satisfies

$$\mathcal{A}(u;\, supp\, \varphi) \leq \mathcal{A}(u + \varphi;\, supp\, \varphi)$$

for every $\varphi \in W^{1,p}_0(\Omega, \mathbb{R}^n)$.
Partial regularity for solutions of nonlinear elliptic systems has been studied by Morrey

"Partial regularity results for nonlinear elliptic systems", J. Math. and Mech., 1969;

Giusti in

"Regolarità parziale delle soluzioni di sistemi ellittici quasi lineari di ordine arbitrario", Ann. Sc. Norm. Sup. Pisa, 1969;

Giusti in

"Regolarità parziale delle soluzioni di sistemi ellittici quasi lineari di ordine arbitrario", Ann. Sc. Norm. Sup. Pisa, 1969;

Giusti and Miranda in

"Sulla regolarità delle soluzioni di una classe di sistemi ellittici quasilineari", Arch. Rat. Mech. Anal., 1968;

Giusti in

“Regolarità parziale delle soluzioni di sistemi ellittici quasi lineari di ordine arbitrario”, Ann. Sc. Norm. Sup. Pisa, 1969;

Giusti and Miranda in

“Sulla regolarità delle soluzioni di una classe di sistemi ellittici quasilineari”, Arch. Rat. Mech. Anal., 1968;

using an **indirect argument**

Giusti in

"Regolarità parziale delle soluzioni di sistemi ellittici quasi lineari di ordine arbitrario", Ann. Sc. Norm. Sup. Pisa, 1969;

Giusti and Miranda in

"Sulla regolarità delle soluzioni di una classe di sistemi ellittici quasilineari", Arch. Rat. Mech. Anal., 1968;

using an **indirect argument** similar to that one introduced by De Giorgi and Almgren in the regularity theory of parametrix minimal surfaces.

Giusti in

"Regolarità parziale delle soluzioni di sistemi ellittici quasi lineari di ordine arbitrario", Ann. Sc. Norm. Sup. Pisa, 1969;

Giusti and Miranda in

"Sulla regolarità delle soluzioni di una classe di sistemi ellittici quasilineari", Arch. Rat. Mech. Anal., 1968;

using an **indirect argument** similar to that one introduced by De Giorgi and Almgren in the regularity theory of parametrix minimal surfaces.
New perturbation arguments

Giusti in

"Regolarità parziale delle soluzioni di sistemi ellittici quasi lineari di ordine arbitrario", Ann. Sc. Norm. Sup. Pisa, 1969;

Giusti and Miranda in

"Sulla regolarità delle soluzioni di una classe di sistemi ellittici quasilineari", Arch. Rat. Mech. Anal., 1968;

using an **indirect argument** similar to that one introduced by De Giorgi and Almgren in the regularity theory of parametrix minimal surfaces.
New perturbation arguments are later considered by Giaquinta and Giusti in

"Partial regularity for the solution to nonlinear parabolic systems", Ann. Mat. Pura Appl. 1973;

Giaquinta and Modica in

"Regularity results for some classes of higher order non linear elliptic systems", J. Reine Angew. Math., 1979;

to study higher integrability of the gradient of the solutions.

Giaquinta and Modica in

"Regularity results for some classes of higher order non linear elliptic systems", J. Reine Angew. Math., 1979;

to study higher integrability of the gradient of the solutions. Using a **perturbation method, or direct argument, A. Tachikawa and R.**

Giaquinta and Modica in

"Regularity results for some classes of higher order non linear elliptic systems", J. Reine Angew. Math., 1979;

to study higher integrability of the gradient of the solutions. Using a **perturbation method, or direct argument, A. Tachikawa and R.** prove partial regularity for minimizers of the following variational integrals

Giaquinta and Modica in

"Regularity results for some classes of higher order non linear elliptic systems", J. Reine Angew. Math., 1979;

to study higher integrability of the gradient of the solutions. Using a **perturbation method, or direct argument, A. Tachikawa and R.** prove partial regularity for minimizers of the following variational integrals

$$\mathcal{A}(u;\Omega) := \int_\Omega F(x,u,Du)dx$$

Giaquinta and Modica in

"Regularity results for some classes of higher order non linear elliptic systems", J. Reine Angew. Math., 1979;

to study higher integrability of the gradient of the solutions. Using a **perturbation method, or direct argument, A. Tachikawa and R.** prove partial regularity for minimizers of the following variational integrals

$$\mathcal{A}(u;\Omega) := \int_\Omega F(x,u,Du)dx$$

where $u:\Omega\to\mathbb{R}^n$, $Du=(D_\alpha u^i)$, $(\alpha=1,\dots,m,\ i=1,\dots,n)$ and the integrand has the following special form

Giaquinta and Modica in

"Regularity results for some classes of higher order non linear elliptic systems", J. Reine Angew. Math., 1979;

to study higher integrability of the gradient of the solutions. Using a **perturbation method, or direct argument, A. Tachikawa and R.** prove partial regularity for minimizers of the following variational integrals

$$\mathcal{A}(u;\Omega) := \int_\Omega F(x,u,Du)dx$$

where $u:\Omega \to \mathbb{R}^n$, $Du = (D_\alpha u^i)$, $(\alpha = 1,\dots,m,\ i = 1,\dots,n)$ and the integrand has the following special form

$$F(x,u,Du) = A(x,u,g^{\alpha\beta}(x)h_{ij}(u)D_\alpha u^i D_\beta u^j).$$

This kind of functionals arises as $p-$**energy of maps between Riemannian manifolds.** From this point of view, the **geometric interest** may occur on the above functionals.

This kind of functionals arises as $p-$**energy of maps between Riemannian manifolds.** From this point of view, the **geometric interest** may occur on the above functionals.
Moreover, some **methods of proofs** of regularity for nonlinear elliptic systems can be applied to the equations of nonlinear **Hodge theory,**

This kind of functionals arises as $p-$**energy of maps between Riemannian manifolds.** From this point of view, the **geometric interest** may occur on the above functionals.
Moreover, some **methods of proofs** of regularity for nonlinear elliptic systems can be applied to the equations of nonlinear **Hodge theory,**
studied e.g. by L. M. Sibner and R. B. Sibner in
"A non-linear Hodge de Rham theorem", Acta Math., 1970.

This kind of functionals arises as $p-$**energy of maps between Riemannian manifolds.** From this point of view, the **geometric interest** may occur on the above functionals.
Moreover, some **methods of proofs** of regularity for nonlinear elliptic systems can be applied to the equations of nonlinear
Hodge theory,
studied e.g. by L. M. Sibner and R. B. Sibner in
"A non-linear Hodge de Rham theorem", Acta Math., 1970.
Also, Giaquinta and Giusti in
"The singular set of the minima of certain quadratic functionals", Ann. Sc. Norm. Sup. Pisa, 1984,

This kind of functionals arises as $p-$**energy of maps between Riemannian manifolds.** From this point of view, the **geometric interest** may occur on the above functionals.
Moreover, some **methods of proofs** of regularity for nonlinear elliptic systems can be applied to the equations of nonlinear **Hodge theory,**
studied e.g. by L. M. Sibner and R. B. Sibner in
"A non-linear Hodge de Rham theorem", Acta Math., 1970.
Also, Giaquinta and Giusti in
"The singular set of the minima of certain quadratic functionals", Ann. Sc. Norm. Sup. Pisa, 1984,
consider the quadratic functionals

$$\int_\Omega g^{\alpha\,\beta}(x)\, h_{ij}(u)\, D_\alpha u^i\, D_\beta u^j\, dx$$

where $g^{\alpha\,\beta}$ and h_{ij} are symmetric positive definite matrices having smooth coefficients.

A geometrically useful **example** is the following

$$\int_{B_1(0)} \frac{|Du|^2}{(1+|u|^2)} dx$$

that is, in local coordinates, the **energy of a map**.

A geometrically useful **example** is the following

$$\int_{B_1(0)} \frac{|Du|^2}{(1+|u|^2)} dx$$

that is, in local coordinates, the **energy of a map**.
Under similar assumptions to that in the previous paper is proved by

Giaquinta and G. Modica in "Partial regularity of minimizers of quasiconvex integrals", Ann. Inst. H. Poincaré 1986,

that minimizers have Hölder continuous derivatives in an open set

$$\Omega_0 \subset \Omega : meas(\Omega \backslash \Omega_0) \; = \; 0.$$

that minimizers have Hölder continuous derivatives in an open set

$$\Omega_0 \subset \Omega : meas(\Omega \backslash \Omega_0) = 0.$$

The hypothesis we consider in this talk has been inspired by Campanato, see

"A maximum principle for non-linear elliptic systems: Boundary fundamental estimates", Adv. Math., 1987,

that minimizers have Hölder continuous derivatives in an open set

$$\Omega_0 \subset \Omega : meas(\Omega \backslash \Omega_0) \ = \ 0.$$

The hypothesis we consider in this talk has been inspired by Campanato, see

"A maximum principle for non-linear elliptic systems: Boundary fundamental estimates", Adv. Math., 1987,

where are obtained deep **Hölder regularity** results in $\mathcal{L}^{p,\lambda}$ spaces

that minimizers have Hölder continuous derivatives in an open set

$$\Omega_0 \subset \Omega : meas(\Omega \backslash \Omega_0) = 0.$$

The hypothesis we consider in this talk has been inspired by Campanato, see

"A maximum principle for non-linear elliptic systems: Boundary fundamental estimates", Adv. Math., 1987,

where are obtained deep **Hölder regularity** results in $\mathcal{L}^{p,\lambda}$ spaces for solutions of elliptic systems **having nonlinearity greater or equal to 2.**

In the previous note the coefficients of the second order elliptic differential operators are supposed **continuous**,

In the previous note the coefficients of the second order elliptic differential operators are supposed **continuous**, the **VMO assumption** is a **more recent idea.**

In the previous note the coefficients of the second order elliptic differential operators are supposed **continuous**, the **VMO assumption** is a **more recent idea.**

A. Tachikawa and R., Journal of London Math. Society, 2005

In the previous note the coefficients of the second order elliptic differential operators are supposed **continuous**, the **VMO assumption** is a **more recent idea.**

A. Tachikawa and R., Journal of London Math. Society, 2005 investigate the partial regularity of the minimizers of quadratic functionals, whose integrands **have** *VMO* **coefficients**,

In the previous note the coefficients of the second order elliptic differential operators are supposed **continuous**, the **VMO assumption** is a **more recent idea.**
A. Tachikawa and R., Journal of London Math. Society, 2005 investigate the partial regularity of the minimizers of quadratic functionals, whose integrands **have** *VMO* **coefficients**, using some majorizations for the functionals, rather than the well known Euler's equation associated to it.

In the previous note the coefficients of the second order elliptic differential operators are supposed **continuous**, the **VMO assumption** is a **more recent idea.**
A. Tachikawa and R., Journal of London Math. Society, 2005 investigate the partial regularity of the minimizers of quadratic functionals, whose integrands **have** *VMO* **coefficients**, using some majorizations for the functionals, rather than the well known Euler's equation associated to it. The functional is

$$\int_\Omega \left\{ A_{ij}^{\alpha\beta}(x,u) D_\alpha u^i D_\beta u^j + g(x,u,Du) \right\} dx,$$

In the previous note the coefficients of the second order elliptic differential operators are supposed **continuous**, the **VMO assumption** is a **more recent idea.**

A. Tachikawa and R., Journal of London Math. Society, 2005 investigate the partial regularity of the minimizers of quadratic functionals, whose integrands **have** *VMO* **coefficients**, using some majorizations for the functionals, rather than the well known Euler's equation associated to it. The functional is

$$\int_\Omega \left\{ A_{ij}^{\alpha\beta}(x,u) D_\alpha u^i D_\beta u^j + g(x,u,Du) \right\} dx,$$

where $\Omega \subset \mathbb{R}^m$, $m \geq 3$, is a bounded open set, $u\colon \Omega \to \mathbb{R}^n$, $n > 1$,

1. $A_{ij}^{\alpha\beta} = A_{ji}^{\beta\alpha}$.

1. $A_{ij}^{\alpha\beta} = A_{ji}^{\beta\alpha}$.
2. For every $u \in \mathbb{R}^n$, $A_{ij}^{\alpha\beta}(\cdot, u) \in VMO(\Omega)$.

We should mention that since C^0 is a **proper subset** of VMO, the **continuity of** $A_{ij}^{\alpha\beta}(x, u)$ with respect to x is **not assumed.**

1. $A_{ij}^{\alpha\beta} = A_{ji}^{\beta\alpha}$.
2. For every $u \in \mathbb{R}^n$, $A_{ij}^{\alpha\beta}(\cdot, u) \in VMO(\Omega)$.
3. For every $x \in \Omega$ and $u, v \in \mathbb{R}^n$,

$$\left|A_{ij}^{\alpha\beta}(x,u) - A_{ij}^{\alpha\beta}(x,v)\right| \leq \omega(|u-v|^2)$$

for some monotone increasing concave function $\omega : \omega(0) = 0$.

We should mention that since C^0 is a **proper subset** of VMO, the **continuity of** $A_{ij}^{\alpha\beta}(x,u)$ with respect to x is **not assumed.**

1. $A_{ij}^{\alpha\beta} = A_{ji}^{\beta\alpha}$.
2. For every $u \in \mathbb{R}^n$, $A_{ij}^{\alpha\beta}(\cdot, u) \in VMO(\Omega)$.
3. For every $x \in \Omega$ and $u, v \in \mathbb{R}^n$,

$$\left|A_{ij}^{\alpha\beta}(x,u) - A_{ij}^{\alpha\beta}(x,v)\right| \leq \omega(|u-v|^2)$$

   for some monotone increasing concave function $\omega : \omega(0) = 0$.
4. There exists $\nu > 0$:

$$\nu|\xi|^2 \leq A_{ij}^{\alpha\beta}(x,u)\xi_\alpha^i \xi_\beta^j$$

   for a.e. $x \in \Omega$, for all $u \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^{mn}$.

We should mention that since C^0 is a **proper subset** of VMO, the **continuity of** $A_{ij}^{\alpha\beta}(x,u)$ with respect to x is **not assumed.**

We suppose that the function g is a Charathéodory function and has growth less than quadratic:

We suppose that the function g is a Charathéodory function and has growth less than quadratic:

(g1) $g(\cdot, u, Du)$ is measurable in x $\forall u \in \mathbb{R}^N, \forall z \in \mathbb{R}^{nN}$;

(g2) $g(x, \cdot, \cdot)$ is continuous in (u, z) $a.\ e. x \in \Omega$.

We suppose that the function g is a Charathéodory function and has growth less than quadratic:

(g1) $g(\cdot, u, Du)$ is measurable in x $\forall u \in \mathbb{R}^N, \forall z \in \mathbb{R}^{nN}$;

(g2) $g(x, \cdot, \cdot)$ is continuous in (u, z) $a.\ e.\ x \in \Omega$.

Moreover, g satisfies the condition:

(g3)

$$|g(x, u, z)| \leq g_1(x) + H\, |z|^{\gamma},$$

$g_1 \geq 0$ a. e. in Ω, $g_1 \in L^p(\Omega)$, $2 < p \leq \infty$, $H \geq 0$, $0 \leq \gamma < 2$.

Theorem (RT London Math. Soc., 2005)

Let $\Omega \subset \mathbb{R}^m, m \geq 3,$ *bounded open set,* $u \in W^{1,2}(\Omega, \mathbb{R}^n)$ *be a minimum of the functional above defined. Suppose that assumptions on* $A_{ij}^{\alpha\beta}(x,u)$ *and* $g(x,u,Du)$ *are satisfied.*

Theorem (RT London Math. Soc., 2005)

Let $\Omega \subset \mathbb{R}^m, m \geq 3$, *bounded open set,* $u \in W^{1,2}(\Omega, \mathbb{R}^n)$ *be a minimum of the functional above defined. Suppose that assumptions on* $A_{ij}^{\alpha\beta}(x, u)$ *and* $g(x, u, Du)$ *are satisfied.*
Then, for $\lambda = m(1 - \frac{2}{p})$, *we have*

$$D\,u \in L^{2,\lambda}_{\mathrm{loc}}(\Omega_0)$$

Theorem (RT London Math. Soc., 2005)

Let $\Omega \subset \mathbb{R}^m, m \geq 3,$ *bounded open set,* $u \in W^{1,2}(\Omega, \mathbb{R}^n)$ *be a minimum of the functional above defined. Suppose that assumptions on* $A_{ij}^{\alpha\beta}(x,u)$ *and* $g(x,u,Du)$ *are satisfied.*
Then, for $\lambda = m(1 - \frac{2}{p})$, *we have*

$$D\,u \in L^{2,\lambda}_{\mathrm{loc}}(\Omega_0)$$

where

$$\Omega_0 = \{x \in \Omega\colon \liminf_{R \to 0} \frac{1}{R^{m-2}} \int_{B(x,R)} |Du(y)|^2 dy = 0\}.$$

Theorem (RT London Math. Soc., 2005)

Let $\Omega \subset \mathbb{R}^m, m \geq 3,$ *bounded open set,* $u \in W^{1,2}(\Omega, \mathbb{R}^n)$ *be a minimum of the functional above defined. Suppose that assumptions on* $A_{ij}^{\alpha\beta}(x,u)$ *and* $g(x,u,Du)$ *are satisfied.*
Then, for $\lambda = m(1-\frac{2}{p})$*, we have*

$$D\,u \in L^{2,\lambda}_{\mathrm{loc}}(\Omega_0)$$

where

$$\Omega_0 = \{x \in \Omega : \liminf_{R\to 0} \frac{1}{R^{m-2}} \int_{B(x,R)} |Du(y)|^2 dy = 0\}.$$

As a consequence, for $\alpha \in (0,1)$,

$$u \in C^{0,\alpha}(\Omega_0).$$

In a subsequent note A. Tachikawa and R., R.I.M.S., Kyoto, 2004 extend the above regularity results because we consider the more general class **Campanato** spaces.

In a subsequent note A. Tachikawa and R., R.I.M.S., Kyoto, 2004 extend the above regularity results because we consider the more general class **Campanato** spaces.
Let $1 \leq p < \infty$ and $\lambda \geq 0$.

In a subsequent note A. Tachikawa and R., R.I.M.S., Kyoto, 2004 extend the above regularity results because we consider the more general class **Campanato** spaces.
Let $1 \leq p < \infty$ and $\lambda \geq 0$.
By **Campanato spaces** $\mathcal{L}^{p,\lambda}(\Omega)$ we denote the linear space of functions $u \in L^p(\Omega)$:

In a subsequent note A. Tachikawa and R., R.I.M.S., Kyoto, 2004 extend the above regularity results because we consider the more general class **Campanato** spaces.

Let $1 \leq p < \infty$ and $\lambda \geq 0$.

By **Campanato spaces** $\mathcal{L}^{p,\lambda}(\Omega)$ we denote the linear space of functions $u \in L^p(\Omega)$:

$$[f]_{p,\lambda} = \left\{ \sup_{\substack{x \in \Omega \\ 0<\rho<\mathrm{diam}\Omega}} \rho^{-\lambda} \int_{\Omega \cap B(x,\rho)} |f(y) - f_\rho|^p dy \right\}^{\frac{1}{p}} < +\infty.$$

$\mathcal{L}^{p,\lambda}(\Omega)$ are **Banach spaces** with the following **norm**

$$\|f\|_{\mathcal{L}^{p,\lambda}(\Omega)} = \|f\|_{L^p(\Omega)} + [f]_{p,\lambda}.$$

$\mathcal{L}^{p,\lambda}(\Omega)$ are **Banach spaces** with the following **norm**

$$\|f\|_{\mathcal{L}^{p,\lambda}(\Omega)} = \|f\|_{L^p(\Omega)} + [f]_{p,\lambda}.$$

A function

$$u \in \mathcal{L}^{p,\lambda}(\Omega)$$

$$\Updownarrow$$

$$\sup_{x\in\Omega, 0<\rho<\operatorname{diam}\Omega} \rho^{-\lambda} \inf_{c\in\mathbb{R}} \int_{\Omega\cap B(x,\rho)} |f - c|^p dy < \infty.$$

$\mathcal{L}^{p,\lambda}(\Omega)$ are **Banach spaces** with the following **norm**

$$\|f\|_{\mathcal{L}^{p,\lambda}(\Omega)} = \|f\|_{L^p(\Omega)} + [f]_{p,\lambda}.$$

A function

$$u \in \mathcal{L}^{p,\lambda}(\Omega)$$

$$\Updownarrow$$

$$\sup_{x\in\Omega, 0<\rho<\operatorname{diam}\Omega} \rho^{-\lambda} \inf_{c\in\mathbb{R}} \int_{\Omega\cap B(x,\rho)} |f - c|^p dy < \infty.$$

Using Hölder inequality :

$$\mathcal{L}^{p_1,\lambda_1}(\Omega) \subset \mathcal{L}^{p,\lambda}(\Omega)$$

$\mathcal{L}^{p,\lambda}(\Omega)$ are **Banach spaces** with the following **norm**

$$\|f\|_{\mathcal{L}^{p,\lambda}(\Omega)} = \|f\|_{L^p(\Omega)} + [f]_{p,\lambda}.$$

A function

$$u \in \mathcal{L}^{p,\lambda}(\Omega)$$

$$\Updownarrow$$

$$\sup_{x\in\Omega, 0<\rho<\operatorname{diam}\Omega} \rho^{-\lambda} \inf_{c\in\mathbb{R}} \int_{\Omega\cap B(x,\rho)} |f - c|^p dy < \infty.$$

Using Hölder inequality :

$$\mathcal{L}^{p_1,\lambda_1}(\Omega) \subset \mathcal{L}^{p,\lambda}(\Omega)$$

where

$$p \le p_1, \qquad \frac{n-\lambda}{p} \ge \frac{n-\lambda_1}{p_1}.$$

For $0 \leq \lambda < m$

$$[f]_{p,\lambda} \leq C\|f\|_{L^{p,\lambda}(\Omega)}$$

For $0 \leq \lambda < m$

$$[f]_{p,\lambda} \leq C\|f\|_{L^{p,\lambda}(\Omega)}$$

relation between **Morrey** and **Campanato spaces**

$$L^{p,\lambda}(\Omega) \subset \mathcal{L}^{p,\lambda}(\Omega).$$

Theorem (RT R.I.M.S., 2004)

Let $\Omega \subset \mathbb{R}^m$, $m \geq 3$, $1 < q \leq 2$, $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ *be a minimum of* $\mathcal{A}(u, \Omega)$ *defined above, satisfying the same assumptions.*

Theorem (RT R.I.M.S., 2004)

Let $\Omega \subset \mathbb{R}^m$, $m \geq 3$, $1 < q \leq 2$, $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ be a minimum of $\mathcal{A}(u, \Omega)$ defined above, satisfying the same assumptions. Then, for $\lambda = m(1 - \frac{q}{p})$ we have

$$D\, u \in \mathcal{L}^{q,\lambda}_{\mathrm{loc}}(\Omega_0, \mathbb{R}^{n\,N})$$

Theorem (RT R.I.M.S., 2004)

Let $\Omega \subset \mathbb{R}^m$, $m \geq 3$, $1 < q \leq 2$, $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ be a minimum of $\mathcal{A}(u, \Omega)$ defined above, satisfying the same assumptions. Then, for $\lambda = m(1 - \frac{q}{p})$ we have

$$D\,u \in \mathcal{L}^{q,\lambda}_{\mathrm{loc}}(\Omega_0, \mathbb{R}^{n\,N})$$

where

$$\Omega_0 = \{\, x \in \Omega\colon \liminf_{R \to 0} \frac{1}{R^{m-2}} \int_{B(x,R)} |Du(y)|^2 dy = 0 \,\}.$$

In both two Theorems we point out the **existence of** Ω_0.

In both two Theorems we point out the **existence of** Ω_0.
The set Ω_0 is obligatory,

In both two Theorems we point out the **existence of** Ω_0.
The set Ω_0 is obligatory, in fact when we pass from the regularity theory for **scalar** minimizers of solutions of elliptic equations

In both two Theorems we point out the **existence of** Ω_0.
The set Ω_0 is obligatory, in fact when we pass from the regularity theory for **scalar** minimizers of solutions of elliptic equations to the regularity theory for **vector-valued** minimizers of solutions of elliptic systems, **the situation changes completely**:

In both two Theorems we point out the **existence of** Ω_0.
The set Ω_0 is obligatory, in fact when we pass from the regularity theory for **scalar** minimizers of solutions of elliptic equations to the regularity theory for **vector-valued** minimizers of solutions of elliptic systems, **the situation changes completely**:
regularity is an **exceptional occurence everywhere**, excluding the two dimensional case.

In both two Theorems we point out the **existence of** Ω_0.
The set Ω_0 is obligatory, in fact when we pass from the regularity theory for **scalar** minimizers of solutions of elliptic equations to the regularity theory for **vector-valued** minimizers of solutions of elliptic systems, **the situation changes completely**:
regularity is an **exceptional occurence everywhere**, excluding the two dimensional case.
In **1968 De Giorgi** showed that his regularity result for solutions of second order elliptic equations with measurable bounded coefficients **cannot be extended** to solutions of elliptic systems.

In both two Theorems we point out the **existence of** Ω_0.
The set Ω_0 is obligatory, in fact when we pass from the regularity theory for **scalar** minimizers of solutions of elliptic equations to the regularity theory for **vector-valued** minimizers of solutions of elliptic systems, **the situation changes completely**:
regularity is an **exceptional occurence everywhere**, excluding the two dimensional case.
In **1968 De Giorgi** showed that his regularity result for solutions of second order elliptic equations with measurable bounded coefficients **cannot be extended** to solutions of elliptic systems.
He presented the quadratic functional

$$\mathcal{S} = \int_\Omega A_{ij}^{\alpha\beta}(x) D_\alpha u^i D_\beta u^j \, dx$$

with $A_{ij}^{\alpha\beta} \in L^\infty(\Omega)$, such that

$$\exists \nu > 0 : A_{ij}^{\alpha\beta} \chi_\alpha^i \chi_\beta^j \geq \nu |\chi|^2, \text{a.e.} x \in \Omega, \forall \chi \in \mathbb{R}^{nN}.$$

De Giorgi proves that $\mathcal{S}$ has a minimizer that is a function having a **point of discontinuity** in the origin.

Later, Souček in

Later, Souček in
"Singular solutions to linear elliptic systems", Comment. Math. Univ. Carolinae, 1984

Later, Souček in
"Singular solutions to linear elliptic systems", Comment. Math. Univ. Carolinae, 1984
showed that minimizers of functionals of $\mathcal{S}$ type can be discontinuous, **not only in a point**, but on a **dense subset of** Ω.

Later, Souček in
"Singular solutions to linear elliptic systems", Comment. Math. Univ. Carolinae, 1984
showed that minimizers of functionals of $\mathcal{S}$ type can be discontinuous, **not only in a point**, but on a **dense subset of** Ω.
Modifying De Giorgi's example, Giusti and Miranda in 1968 showed that solutions of the quasilinear elliptic systems

$$\int_\Omega A_{ij}^{\alpha\beta}(u) D_\alpha u^i D_\beta \varphi^j \, dx = 0, \forall \varphi \in C_0^\infty(\Omega, \mathbb{R}^N)$$

Later, Souček in
"Singular solutions to linear elliptic systems", Comment. Math. Univ. Carolinae, 1984
showed that minimizers of functionals of $\mathcal{S}$ type can be discontinuous, **not only in a point**, but on a **dense subset of** Ω.
Modifying De Giorgi's example, Giusti and Miranda in 1968 showed that solutions of the quasilinear elliptic systems

$$\int_\Omega A_{ij}^{\alpha\beta}(u) D_\alpha u^i D_\beta \varphi^j \, dx = 0, \forall \varphi \in C_0^\infty(\Omega, \mathbb{R}^N)$$

with analytic elliptic coefficients $A_{ij}^{\alpha\beta}$, have **singularities in dimension** $m \geq 3$.

Similar examples were presented in the meantime independently by **Maz'ya**.

Similar examples were presented in the meantime independently by **Maz'ya**. **Giaquinta**, Similar examples were presented in the meantime independently by **Maz'ya**. **Giaquinta**, **Morrey** and others

Similar examples were presented in the meantime independently by **Maz'ya**. **Giaquinta**, **Morrey** and others were interested in these problems of solutions of elliptic systems in general **non regular**.

Similar examples were presented in the meantime independently by **Maz'ya**. **Giaquinta**, **Morrey** and others were interested in these problems of solutions of elliptic systems in general **non regular**. Then, we can prove regularity **except on a set**, Similar examples were presented in the meantime independently by **Maz'ya**. **Giaquinta**, **Morrey** and others were interested in these problems of solutions of elliptic systems in general **non regular**. Then, we can prove regularity **except on a set**, hopefully

not too large.

We recall that, for linear systems, regularity results assuming $A_{ij}^{\alpha\beta}$ **constants** or **in** $C^0(\overline{\Omega})$, have been obtained by Campanato in Quaderni S.N.S. Pisa, 1980.

We recall that, for linear systems, regularity results assuming $A_{ij}^{\alpha\beta}$ **constants** or **in** $C^0(\overline{\Omega})$, have been obtained by Campanato in Quaderni S.N.S. Pisa, 1980. **Without** assuming **continuity** of coefficients Acquistapace :

We recall that, for linear systems, regularity results assuming $A_{ij}^{\alpha\beta}$ **constants** or **in** $C^0(\overline{\Omega})$, have been obtained by Campanato in Quaderni S.N.S. Pisa, 1980. **Without** assuming **continuity** of coefficients Acquistapace :

“On BMO regularity for linear elliptic systems”, Annali di Matematica 1992,

We recall that, for linear systems, regularity results assuming $A_{ij}^{\alpha\beta}$ **constants** or **in** $C^0(\overline{\Omega})$, have been obtained by Campanato in Quaderni S.N.S. Pisa, 1980. **Without** assuming **continuity** of coefficients Acquistapace :

"On BMO regularity for linear elliptic systems", Annali di Matematica 1992,

refines Campanato's results considering that coefficients $A_{ij}^{\alpha\beta}$ belong to a class **neither**

We recall that, for linear systems, regularity results assuming $A_{ij}^{\alpha\beta}$ **constants** or **in** $C^0(\overline{\Omega})$, have been obtained by Campanato in Quaderni S.N.S. Pisa, 1980. **Without** assuming **continuity** of coefficients Acquistapace :

"On BMO regularity for linear elliptic systems", Annali di Matematica 1992,

refines Campanato's results considering that coefficients $A_{ij}^{\alpha\beta}$ belong to a class **neither** containing **nor** contained

We recall that, for linear systems, regularity results assuming $A_{ij}^{\alpha\beta}$ **constants** or **in** $C^0(\overline{\Omega})$, have been obtained by Campanato in Quaderni S.N.S. Pisa, 1980. **Without** assuming **continuity** of coefficients Acquistapace :

"On BMO regularity for linear elliptic systems", Annali di Matematica 1992,

refines Campanato's results considering that coefficients $A_{ij}^{\alpha\beta}$ belong to a class **neither** containing **nor** contained in $C^0(\overline{\Omega})$,

We recall that, for linear systems, regularity results assuming $A_{ij}^{\alpha\beta}$ **constants** or **in** $C^0(\overline{\Omega})$, have been obtained by Campanato in Quaderni S.N.S. Pisa, 1980. **Without** assuming **continuity** of coefficients Acquistapace :

"On BMO regularity for linear elliptic systems", Annali di Matematica 1992,

refines Campanato's results considering that coefficients $A_{ij}^{\alpha\beta}$ belong to a class **neither** containing **nor** contained in $C^0(\overline{\Omega})$, hence in general **discontinuous**.

We recall that, for linear systems, regularity results assuming $A_{ij}^{\alpha\beta}$ **constants** or **in** $C^0(\overline{\Omega})$, have been obtained by Campanato in Quaderni S.N.S. Pisa, 1980. **Without** assuming **continuity** of coefficients Acquistapace :

"On BMO regularity for linear elliptic systems", Annali di Matematica 1992,

refines Campanato's results considering that coefficients $A_{ij}^{\alpha\beta}$ belong to a class **neither** containing **nor** contained in $C^0(\overline{\Omega})$, hence in general **discontinuous**.
Moreover, Huang :

We recall that, for linear systems, regularity results assuming $A_{ij}^{\alpha\beta}$ **constants** or **in** $C^0(\overline{\Omega})$, have been obtained by Campanato in Quaderni S.N.S. Pisa, 1980. **Without** assuming **continuity** of coefficients Acquistapace :

"On BMO regularity for linear elliptic systems", Annali di Matematica 1992,

refines Campanato's results considering that coefficients $A_{ij}^{\alpha\beta}$ belong to a class **neither** containing **nor** contained in $C^0(\overline{\Omega})$, hence in general **discontinuous**.
Moreover, Huang :

"Estimates on the generalized Morrey spaces $L_{\phi}^{2,\lambda}$ and BMO_{ψ} for linear elliptic systems", Indiana U. M. J., 1996

linear elliptic systems with coefficients in the class **VMO.**

Therefore, it seems to be natural to expect partial regularity results under the condition that the coefficients of the principal terms $A_{ij}^{\alpha\beta} \in$ **VMO**, even for **nonlinear cases**.

Therefore, it seems to be natural to expect partial regularity results under the condition that the coefficients of the principal terms $A_{ij}^{\alpha\beta} \in$ **VMO**, even for **nonlinear cases**.

Daněček and Viszus in

Therefore, it seems to be natural to expect partial regularity results under the condition that the coefficients of the principal terms $A_{ij}^{\alpha\beta} \in$ **VMO**, even for **nonlinear cases**.
Daněček and Viszus in

"$L^{2,\lambda}$-regularity for minima of variational integrals", Boll. Un. Mat. Ital., 2003,

treated the regularity of minimizer for the functional

$$\int_\Omega \left\{ A_{ij}^{\alpha\beta}(\mathbf{x}) D_\alpha u^i D_\beta u^j + g(x, u, Du) \right\} dx,$$

Therefore, it seems to be natural to expect partial regularity results under the condition that the coefficients of the principal terms $A_{ij}^{\alpha\beta} \in$ **VMO**, even for **nonlinear cases**.
Daněček and Viszus in

"$L^{2,\lambda}$-regularity for minima of variational integrals", Boll. Un. Mat. Ital., 2003,

treated the regularity of minimizer for the functional

$$\int_\Omega \left\{ A_{ij}^{\alpha\beta}(\mathbf{x}) D_\alpha u^i D_\beta u^j + g(x, u, Du) \right\} dx,$$

where $g(x, u, Du)$ is a lower order term which satisfies

$$|g(x, u, z)| \leq f(x) + L|z|^\gamma,$$

Therefore, it seems to be natural to expect partial regularity results under the condition that the coefficients of the principal terms $A_{ij}^{\alpha\beta} \in$ **VMO**, even for **nonlinear cases**.
Daněček and Viszus in

"$L^{2,\lambda}$-regularity for minima of variational integrals", Boll. Un. Mat. Ital., 2003,

treated the regularity of minimizer for the functional

$$\int_\Omega \left\{ A_{ij}^{\alpha\beta}(\mathbf{x}) D_\alpha u^i D_\beta u^j + g(x, u, Du) \right\} dx,$$

where $g(x, u, Du)$ is a lower order term which satisfies

$$|g(x, u, z)| \leq f(x) + L|z|^\gamma,$$

being $f \in L^p(\Omega), 2 < p \leq \infty$, $f \geq 0$ a.e. on Ω, $L \geq 0$, $0 \leq \gamma < 2$.

They obtain hölder regularity of minimizer assuming that $A_{ij}^{\alpha\beta}(x) \in VMO$.

They obtain hölder regularity of minimizer assuming that $A_{ij}^{\alpha\beta}(x) \in VMO$.
A. Tachikawa and R. in [London Math Soc, 2005] have **extended both** the results by **Huang** and **Daněček and Viszus**

They obtain hölder regularity of minimizer assuming that $A_{ij}^{\alpha\beta}(x) \in VMO$.
A. Tachikawa and R. in [London Math Soc, 2005] have **extended both** the results by **Huang** and **Daněček and Viszus** because they treat the functional whose integrand contains $g(x, u, Du)$ and has coefficients $A_{ij}^{\alpha\beta}$ dependent not only on x but also on u.

Another step in the walk with A. Tachikawa is the **generalization of ([RT London Math. Soc., 2005])**

Another step in the walk with A. Tachikawa is the **generalization of ([RT London Math. Soc., 2005])** proving $L^{2,\lambda}-$ regularity of minimizers for quadratic functionals whose integrand has a more generalized form depending on x, u, Du, in the case

$$m \leq 4.$$

Another step in the walk with A. Tachikawa is the **generalization of ([RT London Math. Soc., 2005])** proving $L^{2,\lambda}-$ regularity of minimizers for quadratic functionals whose integrand has a more generalized form depending on x, u, Du, in the case

$$m \leq 4.$$

These results are contained in the paper
A. Tachikawa and R. J. Math. Soc. Japan, 2005
where, we set $\Omega \subset \mathbb{R}^m$ a domain,

$$\mathcal{A}(u, \Omega) = \int_\Omega A(x, u, Du)dx,$$

being $A(x, u, p)$ such hat

(A-1) For every $(u, p) \in \mathbb{R}^n \times \mathbb{R}^{mn}$, $A(\cdot, u, p) \in VMO(\Omega)$ and the mean oscillation of $A(\cdot, u, p)/|p|^2$ vanishes uniformly with respect to u, p in the following sense:

(A-1) For every $(u,p) \in \mathbb{R}^n \times \mathbb{R}^{mn}$, $A(\cdot,u,p) \in VMO(\Omega)$ and the mean oscillation of $A(\cdot,u,p)/|p|^2$ vanishes uniformly with respect to u, p in the following sense:
for some function $\sigma(y,\rho) \geq 0$ with

$$\lim_{R\to 0} \sup_{\rho<R} \fint_{B(x,\rho)} \sigma(y,\rho)dy = 0,$$

$A(\cdot,u,p)$ satisfies

(A-1) For every $(u, p) \in \mathbb{R}^n \times \mathbb{R}^{mn}$, $A(\cdot, u, p) \in VMO(\Omega)$ and the mean oscillation of $A(\cdot, u, p)/|p|^2$ vanishes uniformly with respect to u, p in the following sense:
for some function $\sigma(y, \rho) \geq 0$ with

$$\lim_{R \to 0} \sup_{\rho < R} \fint_{B(x,\rho)} \sigma(y, \rho) dy = 0,$$

$A(\cdot, u, p)$ satisfies

$$\left| A(y, u, p) - A_{x,\rho}(u, p) \right| \leq \sigma(y, \rho) |p|^2, \qquad \forall (u, p) \in \mathbb{R}^n \times \mathbb{R}^{mn}$$

(A-1) For every $(u,p) \in \mathbb{R}^n \times \mathbb{R}^{mn}$, $A(\cdot,u,p) \in VMO(\Omega)$ and the mean oscillation of $A(\cdot,u,p)/|p|^2$ vanishes uniformly with respect to u, p in the following sense:
for some function $\sigma(y,\rho) \geq 0$ with

$$\lim_{R\to 0} \sup_{\rho<R} \fint_{B(x,\rho)} \sigma(y,\rho)dy = 0,$$

$A(\cdot,u,p)$ satisfies

$$\left|A(y,u,p) - A_{x,\rho}(u,p)\right| \leq \sigma(y,\rho)|p|^2, \qquad \forall (u,p) \in \mathbb{R}^n \times \mathbb{R}^{mn}$$

(A-2) For every $x \in \Omega$, $p \in \mathbb{R}^{mn}$ and $u, v \in \mathbb{R}^n$,

$$\left|A(x,u,p) - A(x,v,p)\right| \leq \omega(|u-v|^2)|p|^2$$

(A-1) For every $(u,p) \in \mathbb{R}^n \times \mathbb{R}^{mn}$, $A(\cdot,u,p) \in VMO(\Omega)$ and the mean oscillation of $A(\cdot,u,p)/|p|^2$ vanishes uniformly with respect to u, p in the following sense:
for some function $\sigma(y,\rho) \geq 0$ with

$$\lim_{R\to 0} \sup_{\rho<R} \fint_{B(x,\rho)} \sigma(y,\rho)dy = 0,$$

$A(\cdot,u,p)$ satisfies

$$\left|A(y,u,p) - A_{x,\rho}(u,p)\right| \leq \sigma(y,\rho)|p|^2, \qquad \forall (u,p) \in \mathbb{R}^n \times \mathbb{R}^{mn}$$

(A-2) For every $x \in \Omega$, $p \in \mathbb{R}^{mn}$ and $u, v \in \mathbb{R}^n$,

$$\left|A(x,u,p) - A(x,v,p)\right| \leq \omega(|u-v|^2)|p|^2$$

for some monotone increasing concave function $\omega : \omega(0) = 0$.

(A-3) For almost all $x \in \Omega$, $\forall u \in \mathbb{R}^n$, $A(x,u,\cdot) \in C^2(\mathbb{R}^{mn})$.

(A-1) For every $(u,p) \in \mathbb{R}^n \times \mathbb{R}^{mn}$, $A(\cdot,u,p) \in VMO(\Omega)$ and the mean oscillation of $A(\cdot,u,p)/|p|^2$ vanishes uniformly with respect to u, p in the following sense:
for some function $\sigma(y,\rho) \geq 0$ with

$$\lim_{R\to 0} \sup_{\rho<R} \fint_{B(x,\rho)} \sigma(y,\rho)dy = 0,$$

$A(\cdot,u,p)$ satisfies

$$\left|A(y,u,p) - A_{x,\rho}(u,p)\right| \leq \sigma(y,\rho)|p|^2, \qquad \forall (u,p) \in \mathbb{R}^n \times \mathbb{R}^{mn}$$

(A-2) For every $x \in \Omega$, $p \in \mathbb{R}^{mn}$ and $u, v \in \mathbb{R}^n$,

$$\left|A(x,u,p) - A(x,v,p)\right| \leq \omega(|u-v|^2)|p|^2$$

for some monotone increasing concave function $\omega : \omega(0) = 0$.

(A-3) For almost all $x \in \Omega$, $\forall u \in \mathbb{R}^n$, $A(x,u,\cdot) \in C^2(\mathbb{R}^{mn})$.

(A-4) There exist $\mu_1 \geq \mu_0 > 0$:

$$\mu_0|p|^2 \leq A(x,u,p) \leq \mu_1|p|^2, \qquad \forall (x,u,p) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{mn}.$$

Theorem (RT J. Math. Soc. Japan, 2005)

Let $u \in W^{1,2}(\Omega, \mathbb{R}^n)$ be a local minimizer of the functional $\mathcal{A}(u, \Omega)$. Suppose $(\mathrm{A}\text{-}1)$-$(\mathrm{A}\text{-}4)$ be true. Then, for every $0 < \lambda < \min\{2 + \varepsilon, m\}$, for some $\varepsilon > 0$, Theorem (RT J. Math. Soc. Japan, 2005)

Let $u \in W^{1,2}(\Omega, \mathbb{R}^n)$ *be a local minimizer of the functional* $\mathcal{A}(u, \Omega)$. *Suppose* (A-1)-(A-4) *be true. Then, for every* $0 < \lambda < \min\{2 + \varepsilon, m\}$, *for some* $\varepsilon > 0$,

$$D\,u \;\in L^{2,\lambda}_{\mathrm{loc}}(\Omega_0, \mathbb{R}^{mn})$$

Theorem (RT J. Math. Soc. Japan, 2005)

Let $u \in W^{1,2}(\Omega, \mathbb{R}^n)$ *be a local minimizer of the functional* $\mathcal{A}(u, \Omega)$. *Suppose* (A-1)-(A-4) *be true. Then, for every* $0 < \lambda < \min\{2 + \varepsilon, m\}$, *for some* $\varepsilon > 0$,

$$D\,u \;\in L^{2,\lambda}_{\mathrm{loc}}(\Omega_0, \mathbb{R}^{mn})$$

where

$$\Omega_0 \;=\; \{x \in \Omega \colon \liminf_{R\to 0} \frac{1}{R^{m-2}} \int_{B(x,R)} |Du(y)|^2 dy = 0\}.$$

Theorem (RT J. Math. Soc. Japan, 2005)

Let $u \in W^{1,2}(\Omega, \mathbb{R}^n)$ *be a local minimizer of the functional* $\mathcal{A}(u, \Omega)$. *Suppose* (A-1)-(A-4) *be true. Then, for every* $0 < \lambda < \min\{2 + \varepsilon, m\}$, *for some* $\varepsilon > 0$,

$$D\,u \in L^{2,\lambda}_{\mathrm{loc}}(\Omega_0, \mathbb{R}^{mn})$$

where

$$\Omega_0 = \{x \in \Omega\colon \liminf_{R \to 0} \frac{1}{R^{m-2}} \int_{B(x,R)} |Du(y)|^2 dy = 0\}.$$

Moreover

Theorem (RT J. Math. Soc. Japan, 2005)

Let $u \in W^{1,2}(\Omega, \mathbb{R}^n)$ be a local minimizer of the functional $\mathcal{A}(u, \Omega)$. Suppose (A-1)-(A-4) be true. Then, for every $0 < \lambda < \min\{2 + \varepsilon, m\}$, for some $\varepsilon > 0$,

$$D\,u \;\in L^{2,\lambda}_{\mathrm{loc}}(\Omega_0, \mathbb{R}^{mn})$$

where

$$\Omega_0 \;=\; \{x \in \Omega\colon\; \liminf_{R\to 0} \frac{1}{R^{m-2}} \int_{B(x,R)} |Du(y)|^2 dy = 0\}.$$

Moreover

$$\mathcal{H}^{m-2-\delta}(\Omega \setminus \Omega_0) = 0$$

for some $\delta > 0$, where $\mathcal{H}^r$ denotes the r-dimensional Hausdorff measure.

The fruitful cooperation with A. Tachikawa allows to obtain some improvements.

The fruitful cooperation with A. Tachikawa allows to obtain some improvements. Precisely, using the perturbation, or direct, argument in [RT, ZAA, 2008] and [RT DCDS, 2011], The fruitful cooperation with A. Tachikawa allows to obtain some improvements. Precisely, using the perturbation, or direct, argument in [RT, ZAA, 2008] and [RT DCDS, 2011], we study partial regularity for minimizers of the following variational integrals

$$\mathcal{A}(u;\Omega) := \int_\Omega F(x,u,Du)dx$$

where Ω is a domain of $\mathbb{R}^m$ and the integrand

$F(x,u,\xi) : \Omega \times \mathbb{R}^n \times \mathbb{R}^{mn}$ is *p-growth*, $p \geq 2$, namely, for some $0 < \lambda_0 < \Lambda_0$ and $\mu \in \mathbb{R}$, F satisfies

$$\lambda_0(\mu^2 + |\xi|^2)^{p/2} \leq F(x,u,\xi) \leq \Lambda_0(\mu^2 + |\xi|^2)^{p/2}$$

for all $(x,u,\xi) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{mn}$.

$F(x,u,\xi) : \Omega \times \mathbb{R}^n \times \mathbb{R}^{mn}$ is *p-growth,* $p \geq 2$, namely, for some $0 < \lambda_0 < \Lambda_0$ and $\mu \in \mathbb{R}$, F satisfies

$$\lambda_0(\mu^2 + |\xi|^2)^{p/2} \leq F(x,u,\xi) \leq \Lambda_0(\mu^2 + |\xi|^2)^{p/2}$$

for all $(x,u,\xi) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{mn}$.
Moreover,

$$\lambda_0(\mu^2 + |\xi|^2)^{p/2}|\eta|^2 \leq \frac{\partial^2 F(x,u,\xi)}{\partial \xi_\beta^j \partial \xi_\alpha^i}\eta_\alpha^i \eta_\beta^j \leq \Lambda_1(\mu^2 + |\xi|^2)^{p/2}|\eta|^2$$

for all $(x,u,\xi,\eta) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{mn} \times R^{mn}$.

If $\mu = 0$ the variational feature of the functional is very different from one for $\mu \neq 0$.

If $\mu = 0$ the variational feature of the functional is very different from one for $\mu \neq 0$. If $\mu = 0$, the Euler-Lagrange equation of the functional is degenerate at a point $x \in \Omega$ where $Du(x) = 0$.

If $\mu = 0$ the variational feature of the functional is very different from one for $\mu \neq 0$. If $\mu = 0$, the Euler-Lagrange equation of the functional is degenerate at a point $x \in \Omega$ where $Du(x) = 0$.
In [RT, ZAA, 2008], the authors proved partial regularity results for the case that $\mu \neq 0$,

If $\mu = 0$ the variational feature of the functional is very different from one for $\mu \neq 0$. If $\mu = 0$, the Euler-Lagrange equation of the functional is degenerate at a point $x \in \Omega$ where $Du(x) = 0$.
In [RT, ZAA, 2008], the authors proved partial regularity results for the case that $\mu \neq 0$, while in [RT DCDS, 2011]

If $\mu = 0$ the variational feature of the functional is very different from one for $\mu \neq 0$. If $\mu = 0$, the Euler-Lagrange equation of the functional is degenerate at a point $x \in \Omega$ where $Du(x) = 0$.
In [RT, ZAA, 2008], the authors proved partial regularity results for the case that $\mu \neq 0$, while in [RT DCDS, 2011] the integrand of the functional has the special form

If $\mu = 0$ the variational feature of the functional is very different from one for $\mu \neq 0$. If $\mu = 0$, the Euler-Lagrange equation of the functional is degenerate at a point $x \in \Omega$ where $Du(x) = 0$.
In [RT, ZAA, 2008], the authors proved partial regularity results for the case that $\mu \neq 0$, while in [RT DCDS, 2011] the integrand of the functional has the special form

$$F(x, u, Du) = A(x, u, g^{\alpha\beta}(x)h_{ij}(u)D_\alpha u^i D_\beta u^j),$$

If $\mu = 0$ the variational feature of the functional is very different from one for $\mu \neq 0$. If $\mu = 0$, the Euler-Lagrange equation of the functional is degenerate at a point $x \in \Omega$ where $Du(x) = 0$.
In [RT, ZAA, 2008], the authors proved partial regularity results for the case that $\mu \neq 0$, while in [RT DCDS, 2011] the integrand of the functional has the special form

$$F(x, u, Du) = A(x, u, g^{\alpha\beta}(x) h_{ij}(u) D_\alpha u^i D_\beta u^j),$$

and is showed a partial regularity result which holds even if $\mu = 0$.

If $\mu = 0$ the variational feature of the functional is very different from one for $\mu \neq 0$. If $\mu = 0$, the Euler-Lagrange equation of the functional is degenerate at a point $x \in \Omega$ where $Du(x) = 0$.
In [RT, ZAA, 2008], the authors proved partial regularity results for the case that $\mu \neq 0$, while in [RT DCDS, 2011] the integrand of the functional has the special form

$$F(x, u, Du) = A(x, u, g^{\alpha\beta}(x) h_{ij}(u) D_\alpha u^i D_\beta u^j),$$

and is showed a partial regularity result which holds even if $\mu = 0$.
Let us now state both the results and give an idea of the proof only for the second case $\mu = 0$ ([RT DCDS, 2011]).

General but non-degenerate case [RT ZAA, 2008].

(F-1) $\exists \Lambda_1 > \lambda_1 > 0$, $\exists \mu \neq 0 : \forall (x, u, \xi, \eta) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{mn} \times \mathbb{R}^{mn}$,

$$\lambda_1(\mu^2 + |\xi|)^p \leq F(x, u, \xi) \leq \Lambda_1(\mu^2 + |\xi|)^p,$$

General but non-degenerate case [RT ZAA, 2008].

(F-1) $\exists \Lambda_1 > \lambda_1 > 0,\ \exists \mu \neq 0 : \forall (x,u,\xi,\eta) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{mn} \times \mathbb{R}^{mn}$,

$$\lambda_1(\mu^2 + |\xi|)^p \leq F(x,u,\xi) \leq \Lambda_1(\mu^2 + |\xi|)^p,$$

$$\lambda_1(\mu + |\xi|^2)^{p/2-1}|\eta|^2 \leq \frac{\partial^2 F(x,u,\xi)}{\partial \xi^i_\alpha \partial \xi^j_\beta}\eta^i_\alpha \eta^j_\beta \leq \Lambda_1(\mu^2 + |\eta|^2)^{p/2-1}|\xi|^2;$$

General but non-degenerate case [RT ZAA, 2008].

(F-1) $\exists \Lambda_1 > \lambda_1 > 0$, $\exists \mu \neq 0 : \forall (x, u, \xi, \eta) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{mn} \times \mathbb{R}^{mn}$,

$$\lambda_1(\mu^2 + |\xi|)^p \leq F(x, u, \xi) \leq \Lambda_1(\mu^2 + |\xi|)^p,$$

$$\lambda_1(\mu+|\xi|^2)^{p/2-1}|\eta|^2 \leq \frac{\partial^2 F(x, u, \xi)}{\partial \xi^i_\alpha \partial \xi^j_\beta} \eta^i_\alpha \eta^j_\beta \leq \Lambda_1(\mu^2+|\eta|^2)^{p/2-1}|\xi|^2;$$

(F-2) for every $(u, \xi) \in \mathbb{R}^n \times \mathbb{R}^{mn}$, $F(\cdot, u, \xi) \in VMO(\Omega)$ and the mean oscillation of $F(\cdot, u, \xi)/(\mu^2 + |\xi|^2)^{2/p}$ vanishes uniformly with respect to u, ξ in the following sense:

General but non-degenerate case [RT ZAA, 2008].

(F-1) $\exists \Lambda_1 > \lambda_1 > 0$, $\exists \mu \neq 0 : \forall (x, u, \xi, \eta) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{mn} \times \mathbb{R}^{mn}$,

$$\lambda_1(\mu^2 + |\xi|)^p \leq F(x, u, \xi) \leq \Lambda_1(\mu^2 + |\xi|)^p,$$

$$\lambda_1(\mu+|\xi|^2)^{p/2-1}|\eta|^2 \leq \frac{\partial^2 F(x,u,\xi)}{\partial \xi^i_\alpha \partial \xi^j_\beta}\eta^i_\alpha \eta^j_\beta \leq \Lambda_1(\mu^2+|\eta|^2)^{p/2-1}|\xi|^2;$$

(F-2) for every $(u, \xi) \in \mathbb{R}^n \times \mathbb{R}^{mn}$, $F(\cdot, u, \xi) \in VMO(\Omega)$ and the mean oscillation of $F(\cdot, u, \xi)/(\mu^2 + |\xi|^2)^{2/p}$ vanishes uniformly with respect to u, ξ in the following sense:
$\exists \rho_0 > 0$, $\exists \sigma(z, \rho) : \mathbb{R}^m \times [0, \rho_0) \to [0, \infty)$ with

$$\lim_{R\to 0} \sup_{\rho<R} \fint_{B(0,\rho)\cap\Omega} \sigma(z, \rho)dz = 0,$$

General but non-degenerate case [RT ZAA, 2008].

(F-1) $\exists \Lambda_1 > \lambda_1 > 0$, $\exists \mu \neq 0 : \forall (x, u, \xi, \eta) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{mn} \times \mathbb{R}^{mn}$,

$$\lambda_1(\mu^2 + |\xi|)^p \leq F(x, u, \xi) \leq \Lambda_1(\mu^2 + |\xi|)^p,$$

$$\lambda_1(\mu+|\xi|^2)^{p/2-1}|\eta|^2 \leq \frac{\partial^2 F(x, u, \xi)}{\partial \xi^i_\alpha \partial \xi^j_\beta} \eta^i_\alpha \eta^j_\beta \leq \Lambda_1(\mu^2+|\eta|^2)^{p/2-1}|\xi|^2;$$

(F-2) for every $(u, \xi) \in \mathbb{R}^n \times \mathbb{R}^{mn}$, $F(\cdot, u, \xi) \in VMO(\Omega)$ and the mean oscillation of $F(\cdot, u, \xi)/(\mu^2 + |\xi|^2)^{2/p}$ vanishes uniformly with respect to u, ξ in the following sense:
$\exists \rho_0 > 0$, $\exists \sigma(z, \rho) : \mathbb{R}^m \times [0, \rho_0) \to [0, \infty)$ with

$$\lim_{R \to 0} \sup_{\rho < R} \fint_{B(0,\rho) \cap \Omega} \sigma(z, \rho) dz = 0,$$

such that $F(\cdot, u, \xi)$ satisfies, $\forall x \in \overline{\Omega}$, $y \in B(x, \rho_0) \cap \Omega$,

General but non-degenerate case [RT ZAA, 2008].

(F-1) $\exists \Lambda_1 > \lambda_1 > 0$, $\exists \mu \neq 0 : \forall (x,u,\xi,\eta) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{mn} \times \mathbb{R}^{mn}$,

$$\lambda_1(\mu^2 + |\xi|)^p \leq F(x,u,\xi) \leq \Lambda_1(\mu^2 + |\xi|)^p,$$

$$\lambda_1(\mu+|\xi|^2)^{p/2-1}|\eta|^2 \leq \frac{\partial^2 F(x,u,\xi)}{\partial \xi^i_\alpha \partial \xi^j_\beta}\eta^i_\alpha \eta^j_\beta \leq \Lambda_1(\mu^2+|\eta|^2)^{p/2-1}|\xi|^2;$$

(F-2) for every $(u,\xi) \in \mathbb{R}^n \times \mathbb{R}^{mn}$, $F(\cdot,u,\xi) \in VMO(\Omega)$ and the mean oscillation of $F(\cdot,u,\xi)/(\mu^2+|\xi|^2)^{2/p}$ vanishes uniformly with respect to u, ξ in the following sense:
$\exists \rho_0 > 0$, $\exists \sigma(z,\rho) : \mathbb{R}^m \times [0,\rho_0) \to [0,\infty)$ with

$$\lim_{R\to 0} \sup_{\rho<R} ⨍_{B(0,\rho)\cap\Omega} \sigma(z,\rho)dz = 0,$$

such that $F(\cdot,u,\xi)$ satisfies, $\forall x \in \overline{\Omega}$, $y \in B(x,\rho_0) \cap \Omega$,

$$\left|F(y,u,\xi) - F_{x,\rho}(u,\xi)\right| \leq \sigma(x-y,\rho)(\mu^2+|\xi|^2)^{p/2}, \forall (u,\xi) \in \mathbb{R}^n \times \mathbb{R}^{mn},$$

(F-3) for every $x \in \Omega$, $\xi \in \mathbb{R}^{mn}$ and $u, v \in \mathbb{R}^n$

$$\left|F(x,u,\xi) - F(x,v,\xi)\right| \le (1+|\xi|^2)^{\frac{p}{2}}\omega(|u-v|^2);$$

(F-3) for every $x \in \Omega$, $\xi \in \mathbb{R}^{mn}$ and $u, v \in \mathbb{R}^n$

$$\big|F(x,u,\xi) - F(x,v,\xi)\big| \le (1 + |\xi|^2)^{\frac{p}{2}} \omega(|u-v|^2);$$

(F-4) for almost all $x \in \Omega$ and all $u \in \mathbb{R}^n$, $F(x,u,\cdot) \in C^2(\mathbb{R}^{mn})$.

Theorem (RT ZAA, 2008)

Assume $p \geq 2$, $\Omega \subset \mathbb{R}^m$ *a bounded domain with sufficiently smooth boundary* $\partial\Omega$,

Theorem (RT ZAA, 2008)

Assume $p \geq 2$, $\Omega \subset \mathbb{R}^m$ *a bounded domain with sufficiently smooth boundary* $\partial\Omega$, $u \in H^{1,p}(\Omega, \mathbb{R}^n)$ *a minimizer of*

$$\mathcal{F}(u, \Omega) = \int_\Omega F(x, u, Du)dx$$

in $X_g(\Omega) = \{u \in H^{1,p}(\Omega) \ : \ u - g \in H_0^{1,p}(\Omega)\}$ *for a given boundary data* $g \in H^{1,s}(\Omega)$, $s > p$. *Let (F-1)-(F-4) be true.*

Theorem (RT ZAA, 2008)

Assume $p \geq 2$, $\Omega \subset \mathbb{R}^m$ *a bounded domain with sufficiently smooth boundary* $\partial\Omega$, $u \in H^{1,p}(\Omega, \mathbb{R}^n)$ *a minimizer of*

$$\mathcal{F}(u, \Omega) = \int_\Omega F(x, u, Du)dx$$

in $X_g(\Omega) = \{u \in H^{1,p}(\Omega) \ : \ u - g \in H_0^{1,p}(\Omega)\}$ *for a given boundary data* $g \in H^{1,s}(\Omega)$, $s > p$. *Let (F-1)-(F-4) be true. Then, for some* $\varepsilon > 0$, $\forall \tau : 0 < \tau < \min\{2 + \varepsilon, m(1 - \frac{p}{s})\}$

Theorem (RT ZAA, 2008)

Assume $p \geq 2$, $\Omega \subset \mathbb{R}^m$ *a bounded domain with sufficiently smooth boundary* $\partial\Omega$, $u \in H^{1,p}(\Omega, \mathbb{R}^n)$ *a minimizer of*

$$\mathcal{F}(u,\Omega) = \int_\Omega F(x,u,Du)dx$$

in $X_g(\Omega) = \{u \in H^{1,p}(\Omega) \ : \ u - g \in H_0^{1,p}(\Omega)\}$ *for a given boundary data* $g \in H^{1,s}(\Omega)$, $s > p$. *Let (F-1)-(F-4) be true. Then, for some* $\varepsilon > 0$, $\forall \tau : 0 < \tau < \min\{2+\varepsilon, m(1-\frac{p}{s})\}$

$$Du \in L^{p,\tau}(\Omega_0, \mathbb{R}^{mn})$$

Theorem (RT ZAA, 2008)

Assume $p \geq 2$, $\Omega \subset \mathbb{R}^m$ *a bounded domain with sufficiently smooth boundary* $\partial\Omega$, $u \in H^{1,p}(\Omega, \mathbb{R}^n)$ *a minimizer of*

$$\mathcal{F}(u,\Omega) = \int_\Omega F(x,u,Du)dx$$

in $X_g(\Omega) = \{u \in H^{1,p}(\Omega) \ : \ u - g \in H_0^{1,p}(\Omega)\}$ *for a given boundary data* $g \in H^{1,s}(\Omega)$, $s > p$. *Let (F-1)-(F-4) be true. Then, for some* $\varepsilon > 0$, $\forall \tau : 0 < \tau < \min\{2+\varepsilon, m(1-\frac{p}{s})\}$

$$Du \ \in L^{p,\tau}(\Omega_0, \mathbb{R}^{mn})$$

where Ω_0 *is a relatively open subset of* $\overline{\Omega}$ *which satisfies*

Theorem (RT ZAA, 2008)

Assume $p \geq 2$, $\Omega \subset \mathbb{R}^m$ *a bounded domain with sufficiently smooth boundary* $\partial\Omega$, $u \in H^{1,p}(\Omega, \mathbb{R}^n)$ *a minimizer of*

$$\mathcal{F}(u, \Omega) = \int_\Omega F(x, u, Du)dx$$

in $X_g(\Omega) = \{u \in H^{1,p}(\Omega) \ : \ u - g \in H_0^{1,p}(\Omega)\}$ *for a given boundary data* $g \in H^{1,s}(\Omega)$, $s > p$. *Let (F-1)-(F-4) be true. Then, for some* $\varepsilon > 0$, $\forall \tau : 0 < \tau < \min\{2 + \varepsilon, m(1 - \frac{p}{s})\}$

$$Du \in L^{p,\tau}(\Omega_0, \mathbb{R}^{mn})$$

where Ω_0 *is a relatively open subset of* $\overline{\Omega}$ *which satisfies*

$$\overline{\Omega} \setminus \Omega_0 = \{x \in \Omega\colon \liminf_{R \to 0} \frac{1}{R^{m-p}} \int_{\Omega \cap B(x,R)} |Du(y)|^p dy > 0\}.$$

Theorem (RT ZAA, 2008)

Assume $p \geq 2$, $\Omega \subset \mathbb{R}^m$ *a bounded domain with sufficiently smooth boundary* $\partial\Omega$, $u \in H^{1,p}(\Omega, \mathbb{R}^n)$ *a minimizer of*

$$\mathcal{F}(u, \Omega) = \int_\Omega F(x, u, Du) dx$$

in $X_g(\Omega) = \{u \in H^{1,p}(\Omega) \; : \; u - g \in H_0^{1,p}(\Omega)\}$ *for a given boundary data* $g \in H^{1,s}(\Omega)$, $s > p$. *Let (F-1)-(F-4) be true. Then, for some* $\varepsilon > 0$, $\forall \tau : 0 < \tau < \min\{2 + \varepsilon, m(1 - \frac{p}{s})\}$

$$Du \in L^{p,\tau}(\Omega_0, \mathbb{R}^{mn})$$

where Ω_0 *is a relatively open subset of* $\overline{\Omega}$ *which satisfies*

$$\overline{\Omega} \setminus \Omega_0 = \{x \in \Omega \colon \liminf_{R \to 0} \frac{1}{R^{m-p}} \int_{\Omega \cap B(x,R)} |Du(y)|^p dy > 0\}.$$

Moreover, $\mathcal{H}^{m-p-\delta}(\overline{\Omega} \setminus \Omega_0) = 0$ *for some* $\delta > 0$.

As a corollary of the above theorem we have the following partial Hölder regularity result.

Corollary

Let g, u and Ω_0 be as in previous Theorem. Assume that $p + 2 \geq m$ and that $s > \max\{m, p\}$.

As a corollary of the above theorem we have the following partial Hölder regularity result.

Corollary

Let g, u and Ω_0 be as in previous Theorem. Assume that $p + 2 \geq m$ and that $s > \max\{m, p\}$. Then, for some $\alpha \in (0, 1)$,

As a corollary of the above theorem we have the following partial Hölder regularity result.

Corollary

Let g, u and Ω_0 be as in previous Theorem. Assume that $p+2 \geq m$ and that $s > \max\{m, p\}$. Then, for some $\alpha \in (0,1)$,

$$u \in C^{0,\alpha}(\Omega_0, \mathbb{R}^n).$$

As a corollary of the above theorem we have the following partial Hölder regularity result.

Corollary

Let g, u and Ω_0 be as in previous Theorem. Assume that $p+2 \geq m$ and that $s > \max\{m, p\}$. Then, for some $\alpha \in (0,1)$,

$$u \in C^{0,\alpha}(\Omega_0, \mathbb{R}^n).$$

Moreover, as a corollary of the proof of the previous Theorem we have the following full-regularity result for the case that F does not depend on u.

Corollary

Assume that F and g satisfy all assumptions of the previous Theorem and that F does not depend on u.

Corollary

Assume that F and g satisfy all assumptions of the previous Theorem and that F does not depend on u.
Let u be a minimizer of $\mathcal{F}$ in the class X_g, then

Corollary

Assume that F and g satisfy all assumptions of the previous Theorem and that F does not depend on u.
Let u be a minimizer of $\mathcal{F}$ in the class X_g, then

$$Du \in L^{p,\tau}(\Omega, \mathbb{R}^{mn}).$$

Corollary

Assume that F and g satisfy all assumptions of the previous Theorem and that F does not depend on u.
Let u be a minimizer of $\mathcal{F}$ in the class X_g, then

$$Du \in L^{p,\tau}(\Omega, \mathbb{R}^{mn}).$$

Moreover, if $p + 2 \geq m$ and $s > \max\{m, p\}$, we have full-Hölder regularity of u

$$u \in C^{0,\alpha}(\overline{\Omega}, \mathbb{R}^{n}).$$

Degenerate case [RT DCDS, 2011].
Let us set $\mu \geq 0, \ p \geq 2$.

Let the integrand function $A(x, u, t)$ be defined on $\Omega \times \mathbb{R}^n \times \mathbb{R}^{mn}$, in the sequel we assume that it satisfies the following assumptions.

(A-1) There exist positive constants $C, \lambda, \Lambda,\ \lambda \leq \Lambda$ such that

$$\lambda(\mu^2 + t)^{\frac{p}{2}} \leq A(x,u,t) \leq \Lambda(\mu^2 + t)^{\frac{p}{2}},$$

$$\lambda(\mu^2 + t)^{\frac{p}{2}-1} \leq |A_t(x,u,t)| \leq \Lambda(\mu^2 + t)^{\frac{p}{2}-1},$$

$$\lambda(\mu^2 + t)^{\frac{p}{2}-2} \leq A_{tt}(x,u,t) \leq \Lambda(\mu^2 + t)^{\frac{p}{2}-2},$$

for all $(x,u,t) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{mn}$.

(A-2) For every $(u, t) \in \mathbb{R}^n \times \mathbb{R}^{mn}$, $A(\cdot, u, t) \in VMO(\Omega)$

(A-2) For every $(u,t) \in \mathbb{R}^n \times \mathbb{R}^{mn}$, $A(\cdot,u,t) \in VMO(\Omega)$ and the mean oscillation of $A(\cdot,u,t)/(\mu^2+|t|)^{(p/2)}$ vanishes uniformly with respect to u,t in the following sense:

(A-2) For every $(u,t) \in \mathbb{R}^n \times \mathbb{R}^{mn}$, $A(\cdot,u,t) \in VMO(\Omega)$ and the mean oscillation of $A(\cdot,u,t)/(\mu^2+|t|)^{(p/2)}$ vanishes uniformly with respect to u,t in the following sense: $\exists \rho_0$ ¿ 0 and a function $\sigma(z,\rho) : \mathbb{R}^m \times [0,\rho_0[\to [0,+\infty[$ with

$$\lim_{r\to 0} \sup_{\rho<r} \fint_{Q(0,\rho)\cap\Omega} \sigma(z,\rho)dz = 0,$$

(A-2) For every $(u,t) \in \mathbb{R}^n \times \mathbb{R}^{mn}$, $A(\cdot,u,t) \in VMO(\Omega)$ and the mean oscillation of $A(\cdot,u,t)/(\mu^2+|t|)^{(p/2)}$ vanishes uniformly with respect to u, t in the following sense:
$\exists\rho_0$ ¿ 0 and a function $\sigma(z,\rho) : \mathbb{R}^m \times [0,\rho_0[\to [0,+\infty[$ with

$$\lim_{r\to 0}\sup_{\rho<r} \fint_{Q(0,\rho)\cap\Omega} \sigma(z,\rho)dz = 0,$$

such that $A(\cdot,u,t)$ satisfies,

$$\left|A(y,u,t) - A_{x,\rho}(u,t)\right| \leq \sigma(x-y,\rho)(\mu^2 + t)^{\frac{p}{2}},$$

for all $(u,t) \in \mathbb{R}^n \times \mathbb{R}^{mn}$, $\forall x \in \Omega$ and $y \in Q(x,\rho_0) \cap \Omega$;

(A-3) For every $x \in \Omega$, $t \in \mathbb{R}^{mn}$ and $u, v \in \mathbb{R}^n$,

$$\left|A(x,u,t) - A(x,v,t)\right| \leq \omega(|u-v|^2)(\mu^2 + t)^{\frac{p}{2}};$$

(A-3) For every $x \in \Omega$, $t \in \mathbb{R}^{mn}$ and $u, v \in \mathbb{R}^n$,

$$\left|A(x,u,t) - A(x,v,t)\right| \leq \omega(|u-v|^2)(\mu^2 + t)^{\frac{p}{2}};$$

(A-4) For almost all $x \in \Omega$ and all $u \in \mathbb{R}^n$. $A(x,u,\cdot) \in C^2(\mathbb{R}^{mn})$, and A_{tt} satisfies, for some $\alpha > 0$,

$$\left|A_{tt}(x,u,t)t - A_{tt}(x,u,s)s\right| \leq c(\mu^2 + t + s)^{\frac{p-2}{2}-\alpha}|t-s|^{\alpha};$$

(A-3) For every $x \in \Omega$, $t \in \mathbb{R}^{mn}$ and $u, v \in \mathbb{R}^n$,

$$\left|A(x,u,t) - A(x,v,t)\right| \leq \omega(|u-v|^2)(\mu^2 + t)^{\frac{p}{2}};$$

(A-4) For almost all $x \in \Omega$ and all $u \in \mathbb{R}^n$. $A(x,u,\cdot) \in C^2(\mathbb{R}^{mn})$, and A_{tt} satisfies, for some $\alpha > 0$,

$$\left|A_{tt}(x,u,t)t - A_{tt}(x,u,s)s\right| \leq c(\mu^2 + t + s)^{\frac{p-2}{2}-\alpha}|t-s|^{\alpha};$$

(A-5) There exist constants λ_0, Λ_0, λ_1, Λ_1, $0 < \lambda_i < \Lambda_i$, $i = 0, 1$:

$$\lambda_0|\zeta|^2 \leq g^{\alpha\beta}(x)\zeta_\alpha\zeta_\beta \leq \Lambda_0|\zeta|^2,$$

(A-3) For every $x \in \Omega$, $t \in \mathbb{R}^{mn}$ and $u, v \in \mathbb{R}^n$,

$$\left|A(x,u,t) - A(x,v,t)\right| \leq \omega(|u-v|^2)(\mu^2 + t)^{\frac{p}{2}};$$

(A-4) For almost all $x \in \Omega$ and all $u \in \mathbb{R}^n$. $A(x,u,\cdot) \in C^2(\mathbb{R}^{mn})$, and A_{tt} satisfies, for some $\alpha > 0$,

$$\left|A_{tt}(x,u,t)t - A_{tt}(x,u,s)s\right| \leq c(\mu^2 + t + s)^{\frac{p-2}{2}-\alpha}|t-s|^{\alpha};$$

(A-5) There exist constants λ_0, Λ_0, λ_1, Λ_1, $0 < \lambda_i < \Lambda_i$, $i = 0, 1$:

$$\lambda_0|\zeta|^2 \leq g^{\alpha\beta}(x)\zeta_\alpha\zeta_\beta \leq \Lambda_0|\zeta|^2,$$

$$\lambda_1|\eta|^2 \leq h_{ij}(u)\eta^i\eta^j \leq |\eta|^2,$$

(A-3) For every $x \in \Omega$, $t \in \mathbb{R}^{mn}$ and $u, v \in \mathbb{R}^n$,

$$\left|A(x,u,t) - A(x,v,t)\right| \leq \omega(|u-v|^2)(\mu^2 + t)^{\frac{p}{2}};$$

(A-4) For almost all $x \in \Omega$ and all $u \in \mathbb{R}^n$. $A(x,u,\cdot) \in C^2(\mathbb{R}^{mn})$, and A_{tt} satisfies, for some $\alpha > 0$,

$$\left|A_{tt}(x,u,t)t - A_{tt}(x,u,s)s\right| \leq c(\mu^2 + t + s)^{\frac{p-2}{2}-\alpha}|t-s|^{\alpha};$$

(A-5) There exist constants λ_0, Λ_0, λ_1, Λ_1, $0 < \lambda_i < \Lambda_i$, $i = 0, 1$:

$$\lambda_0|\zeta|^2 \leq g^{\alpha\beta}(x)\zeta_\alpha\zeta_\beta \leq \Lambda_0|\zeta|^2,$$

$$\lambda_1|\eta|^2 \leq h_{ij}(u)\eta^i\eta^j \leq |\eta|^2,$$

for all $x \in \Omega$, $u, \zeta \in \mathbb{R}^m$ and $\eta \in \mathbb{R}^n$;

(A-6) For every $u, v \in \mathbb{R}^n$

$$|\, h(u) - h(v) \,| \leq \omega(|\, u - v \,|^2);$$

(A-6) For every $u, v \in \mathbb{R}^n$

$$| h(u) - h(v) | \leq \omega(| u - v |^2);$$

(A-7) g is in the class $L^\infty \cap VMO(\Omega)$.

Theorem (RT DCDS, 2011)

Let $\Omega \subset \mathbb{R}^m$ *be a bounded domain with sufficiently smooth boundary* $\partial\Omega$*. Let also* $u \in W^{1,p}(\Omega, \mathbb{R}^n)$, $p \geq 2$, *be a minimizer of the functional*

$$\mathcal{A}(u, \Omega) = \int_\Omega F(x, u, Du)\, dx$$

Theorem (RT DCDS, 2011)

Let $\Omega \subset \mathbb{R}^m$ *be a bounded domain with sufficiently smooth boundary* $\partial\Omega$*. Let also* $u \in W^{1,p}(\Omega, \mathbb{R}^n)$, $p \geq 2$, *be a minimizer of the functional*

$$\mathcal{A}(u,\Omega) = \int_\Omega F(x,u,Du)\,dx$$

being

$$F(x,u,Du) = A(x,u,g^{\alpha\beta}(x)h_{ij}(u)D_\alpha u^i D_\beta u^j)\,.$$

Theorem (RT DCDS, 2011)

Let $\Omega \subset \mathbb{R}^m$ *be a bounded domain with sufficiently smooth boundary* $\partial\Omega$. *Let also* $u \in W^{1,p}(\Omega, \mathbb{R}^n)$, $p \geq 2$, *be a minimizer of the functional*

$$\mathcal{A}(u, \Omega) = \int_\Omega F(x, u, Du)\, dx$$

being

$$F(x, u, Du) = A(x, u, g^{\alpha\beta}(x) h_{ij}(u) D_\alpha u^i D_\beta u^j) \,.$$

Suppose that $A(x, u, Du)$ *satisfies* (A-1) – (A-7) .

Theorem (RT DCDS, 2011)

Let $\Omega \subset \mathbb{R}^m$ *be a bounded domain with sufficiently smooth boundary* $\partial\Omega$. *Let also* $u \in W^{1,p}(\Omega, \mathbb{R}^n)$, $p \geq 2$, *be a minimizer of the functional*

$$\mathcal{A}(u, \Omega) = \int_\Omega F(x, u, Du)\, dx$$

being

$$F(x, u, Du) = A(x, u, g^{\alpha\beta}(x) h_{ij}(u) D_\alpha u^i D_\beta u^j)\,.$$

Suppose that $A(x, u, Du)$ *satisfies* $(\mathrm{A}\text{-}1) - (\mathrm{A}\text{-}7)$ *.*
Then, there exists an open set $\Omega_0 \subset \Omega$:

Theorem (RT DCDS, 2011)

Let $\Omega \subset \mathbb{R}^m$ be a bounded domain with sufficiently smooth boundary $\partial\Omega$. Let also $u \in W^{1,p}(\Omega, \mathbb{R}^n)$, $p \geq 2$, be a minimizer of the functional

$$\mathcal{A}(u,\Omega) = \int_\Omega F(x,u,Du)\,dx$$

being

$$F(x,u,Du) = A(x,u,g^{\alpha\beta}(x)h_{ij}(u)D_\alpha u^i D_\beta u^j)\,.$$

Suppose that $A(x,u,Du)$ satisfies $(\mathrm{A}\text{-}1) - (\mathrm{A}\text{-}7)$.
Then, there exists an open set $\Omega_0 \subset \Omega$:

$$u \in C^{0,\alpha}(\Omega_0), \quad \forall \alpha \in (0,1).$$

Let us now give and idea of the proof ([RT DCDS, 2011]).

Let us now give and idea of the proof ([RT DCDS, 2011]).
The theorem is proved proceeding as in the note by Giaquinta and Modica "Remarks on the regularity of the minimizers of certain degenerate functionals", Manuscripta Math., 1986.

Let us now give and idea of the proof ([RT DCDS, 2011]).
The theorem is proved proceeding as in the note by Giaquinta and Modica "Remarks on the regularity of the minimizers of certain degenerate functionals", Manuscripta Math., 1986.
Let us set $x_0 \in \Omega$, $R > 0$, $Q(2R) = Q(x_0, 2R) \subset\subset \Omega$.

Let us now give and idea of the proof ([RT DCDS, 2011]).
The theorem is proved proceeding as in the note by Giaquinta and Modica "Remarks on the regularity of the minimizers of certain degenerate functionals", Manuscripta Math., 1986.
Let us set $x_0 \in \Omega$, $R > 0$, $Q(2R) = Q(x_0, 2R) \subset\subset \Omega$.
For every $(u, t) \in \mathbb{R}^n \times \mathbb{R}^{mn}$ we define

Let us now give and idea of the proof ([RT DCDS, 2011]).
The theorem is proved proceeding as in the note by Giaquinta and Modica "Remarks on the regularity of the minimizers of certain degenerate functionals", Manuscripta Math., 1986.
Let us set $x_0 \in \Omega$, $R > 0$, $Q(2R) = Q(x_0, 2R) \subset\subset \Omega$.
For every $(u, t) \in \mathbb{R}^n \times \mathbb{R}^{mn}$ we define

$$A_R(u,t) = \fint_{Q(R)} A(y,u,t)dy, \; u_R = \fint_{Q(R)} u(y)dy, \; g_R = \fint_{Q(R)} g(y)\,dy,$$

and

$$A_0(\zeta) = A_R(u_R, g_R\, h(u_R)\,\zeta\,\zeta)\,.$$

Let us now give and idea of the proof ([RT DCDS, 2011]).
The theorem is proved proceeding as in the note by Giaquinta and Modica "Remarks on the regularity of the minimizers of certain degenerate functionals", Manuscripta Math., 1986.
Let us set $x_0 \in \Omega$, $R > 0$, $Q(2R) = Q(x_0, 2R) \subset\subset \Omega$.
For every $(u, t) \in \mathbb{R}^n \times \mathbb{R}^{mn}$ we define

$$A_R(u,t) = \fint_{Q(R)} A(y,u,t)dy, \; u_R = \fint_{Q(R)} u(y)dy, \; g_R = \fint_{Q(R)} g(y)\,dy,$$

and

$$A_0(\zeta) = A_R(u_R, g_R\, h(u_R)\, \zeta\, \zeta)\,.$$

Now, let us consider the following "frozen functional"

Let us now give and idea of the proof ([RT DCDS, 2011]).
The theorem is proved proceeding as in the note by Giaquinta and Modica "Remarks on the regularity of the minimizers of certain degenerate functionals", Manuscripta Math., 1986.
Let us set $x_0 \in \Omega$, $R > 0$, $Q(2R) = Q(x_0, 2R) \subset\subset \Omega$.
For every $(u, t) \in \mathbb{R}^n \times \mathbb{R}^{mn}$ we define

$$A_R(u,t) = \fint_{Q(R)} A(y,u,t)dy, \; u_R = \fint_{Q(R)} u(y)dy, \; g_R = \fint_{Q(R)} g(y)\,dy,$$

and

$$A_0(\zeta) \;=\; A_R(u_R, g_R\, h(u_R)\,\zeta\,\zeta)\,.$$

Now, let us consider the following "frozen functional"

$$\mathcal{A}_0(u) = \int_{Q(R)} A_0(Du)dx = \int_{Q(R)} A_R(u_R, g_R h(u_R) DuDu)dx.$$

Let also $v \in H^{1,p}(Q(R))$ be a minimizer of $\mathcal{A}_0(\mathcal{V}, Q(R))$ in the set of functions

$$\{\mathcal{V} \in H^{1,p}(Q(R))\ ;\ u - \mathcal{V} \in H_0^{1,p}(Q(R))\}$$

Let also $v \in H^{1,p}(Q(R))$ be a minimizer of $\mathcal{A}_0(\mathcal{V}, Q(R))$ in the set of functions

$$\{\mathcal{V} \in H^{1,p}(Q(R)) \ ; \ u - \mathcal{V} \in H_0^{1,p}(Q(R))\}$$

and $w = u - v$.

Let also $v \in H^{1,p}(Q(R))$ be a minimizer of $\mathcal{A}_0(\mathcal{V}, Q(R))$ in the set of functions

$$\{\mathcal{V} \in H^{1,p}(Q(R)) \; ; \; u - \mathcal{V} \in H_0^{1,p}(Q(R))\}$$

and $w = u - v$.
Moreover, as in the paper by Giaquinta and Modica in Manuscripta Math., 1986, we put

$$H(\xi) = (\mu^2 + |\xi|^2)^{\frac{p}{2}}.$$

Let also $v \in H^{1,p}(Q(R))$ be a minimizer of $\mathcal{A}_0(\mathcal{V}, Q(R))$ in the set of functions

$$\{\mathcal{V} \in H^{1,p}(Q(R)) \ ; \ u - \mathcal{V} \in H_0^{1,p}(Q(R))\}$$

and $w = u - v$.
Moreover, as in the paper by Giaquinta and Modica in Manuscripta Math., 1986, we put

$$H(\xi) = (\mu^2 + |\xi|^2)^{\frac{p}{2}}.$$

In the sequel we use a regularity theorem by Uhlelnbeck, (Acta Math. 1977), for minimizers of functionals of the form

$$\mathcal{F}(v) = \int F(Dv)dx$$

Let also $v \in H^{1,p}(Q(R))$ be a minimizer of $\mathcal{A}_0(\mathcal{V}, Q(R))$ in the set of functions

$$\{\mathcal{V} \in H^{1,p}(Q(R))\ ;\ u - \mathcal{V} \in H_0^{1,p}(Q(R))\}$$

and $w = u - v$.
Moreover, as in the paper by Giaquinta and Modica in Manuscripta Math., 1986, we put

$$H(\xi) = (\mu^2 + |\xi|^2)^{\frac{p}{2}}.$$

In the sequel we use a regularity theorem by Uhlelnbeck, (Acta Math. 1977), for minimizers of functionals of the form

$$\mathcal{F}(v) = \int F(Dv)dx$$

according to it, for $r < \frac{R}{2}$, we have :

$$\int_{Q(R)} H(Dv)dx \leq c\left(\frac{r}{R}\right)^m \int_{Q_{R/2}} H(Dv)dx$$

where c does not depend on r, R, x_0.

Using a formula contained in the note by Giaquinta and Modica in Manusc. Math. 1986 , we have

$$\int_{Q(R)} |Dw|^p dx \leq \{\mathcal{A}_0(u) - \mathcal{A}_0(v)\}$$

Using a formula contained in the note by Giaquinta and Modica in Manusc. Math. 1986 , we have

$$\int_{Q(R)} |Dw|^p dx \leq \{\mathcal{A}_0(u) - \mathcal{A}_0(v)\} =$$

$$= \int_{Q(R)} \left[A_R(u_R, g_R h(u_R) Du Du) - A_R(u_R, g_R h(u_R) Dv Dv) \right] dx.$$

Using a formula contained in the note by Giaquinta and Modica in Manusc. Math. 1986 , we have

$$\int_{Q(R)} |Dw|^p dx \leq \{\mathcal{A}_0(u) - \mathcal{A}_0(v)\} =$$

$$= \int_{Q(R)} \left[A_R(u_R, g_R h(u_R) DuDu) - A_R(u_R, g_R h(u_R) DvDv)\right] dx.$$

Adding and subtracting the terms

$$A(x, u_R, g_R\, h(u_R) DuDu),\ A(x, u, g_R\, h(u_R) DuDu)$$

Using a formula contained in the note by Giaquinta and Modica in Manusc. Math. 1986 , we have

$$\int_{Q(R)} |Dw|^p dx \leq \{\mathcal{A}_0(u) - \mathcal{A}_0(v)\} =$$

$$= \int_{Q(R)} \left[A_R(u_R, g_R h(u_R) Du Du) - A_R(u_R, g_R h(u_R) Dv Dv) \right] dx.$$

Adding and subtracting the terms

$$A(x, u_R, g_R\, h(u_R) Du Du),\; A(x, u, g_R\, h(u_R) Du Du)$$

$$A(x, u, g(x)\, h(u_R) Du Du),\; A(x, u, g(x)\, h(u) Du Du)$$

Using a formula contained in the note by Giaquinta and Modica in Manusc. Math. 1986 , we have

$$\int_{Q(R)} |Dw|^p dx \leq \left\{ \mathcal{A}_0(u) - \mathcal{A}_0(v) \right\} =$$

$$= \int_{Q(R)} \left[A_R(u_R, g_R h(u_R) Du Du) - A_R(u_R, g_R h(u_R) Dv Dv) \right] dx.$$

Adding and subtracting the terms

$$A(x, u_R, g_R\, h(u_R) Du Du), \; A(x, u, g_R\, h(u_R) Du Du)$$

$$A(x, u, g(x)\, h(u_R) Du Du), \; A(x, u, g(x)\, h(u) Du Du)$$

$$A(x, u, g(x)\, h(v) Dv Dv), \; A(x, v, g(x)\, h(v) Dv\, Dv)$$

Using a formula contained in the note by Giaquinta and Modica in Manusc. Math. 1986 , we have

$$\int_{Q(R)} |Dw|^p dx \le \{\mathcal{A}_0(u) - \mathcal{A}_0(v)\} =$$

$$= \int_{Q(R)} \left[A_R(u_R, g_R h(u_R) Du Du) - A_R(u_R, g_R h(u_R) Dv Dv)\right] dx.$$

Adding and subtracting the terms

$$A(x, u_R, g_R\, h(u_R)DuDu),\ A(x, u, g_R\, h(u_R)DuDu)$$

$$A(x, u, g(x)\, h(u_R)DuDu),\ A(x, u, g(x)\, h(u)DuDu)$$

$$A(x, u, g(x)\, h(v)DvDv),\ A(x, v, g(x)\, h(v)Dv\, Dv)$$

$$A(x, v, g(x)\, h(u_R)DvDv),\ A(x, v, g_R\, h(u_R)DvDv)$$

Using a formula contained in the note by Giaquinta and Modica in Manusc. Math. 1986 , we have

$$\int_{Q(R)} |Dw|^p dx \leq \{\mathcal{A}_0(u) - \mathcal{A}_0(v)\} =$$

$$= \int_{Q(R)} \left[A_R(u_R, g_R h(u_R) Du Du) - A_R(u_R, g_R h(u_R) Dv Dv)\right] dx.$$

Adding and subtracting the terms

$$A(x, u_R, g_R\, h(u_R) Du Du),\ A(x, u, g_R\, h(u_R) Du Du)$$

$$A(x, u, g(x)\, h(u_R) Du Du),\ A(x, u, g(x)\, h(u) Du Du)$$

$$A(x, u, g(x)\, h(v) Dv Dv),\ A(x, v, g(x)\, h(v) Dv\, Dv)$$

$$A(x, v, g(x)\, h(u_R) Dv Dv),\ A(x, v, g_R\, h(u_R) Dv Dv)$$

we obtain four kind of integrals.

We obtain, using the assumptions on A :

We obtain, using the assumptions on A :

$$\int_{Q(R)} |Dw|^p \, dx \leq$$

We obtain, using the assumptions on A :

$$\int_{Q(R)} |Dw|^p \, dx \leq c \int_{Q(R)} H(Du) dx \left[\left(\fint_{Q(R)} \sigma(x - x_0, R)^{q'} \, dx \right)^{\frac{1}{q'}} \right.$$

$$+ \left(\fint_{Q(R)} \omega(|u_R - u|^2)^{q'} \, dx \right)^{\frac{1}{q'}} + \left(\fint_{Q(R)} \omega(|u_R - v|^2)^{q'} \, dx \right)^{\frac{1}{q'}}$$

$$\left. + \left(\fint_{Q(R)} |g_R - g(x)|^{q'} \, dx \right)^{\frac{1}{q'}} \right]$$

We obtain, using the assumptions on A :

$$\int_{Q(R)} |Dw|^p \, dx \le c \int_{Q(R)} H(Du) dx \left[\left(\fint_{Q(R)} \sigma(x - x_0, R)^{q'} \, dx \right)^{\frac{1}{q'}} \right.$$

$$+ \left(\fint_{Q(R)} \omega(|u_R - u|^2)^{q'} \, dx \right)^{\frac{1}{q'}} + \left(\fint_{Q(R)} \omega(|u_R - v|^2)^{q'} \, dx \right)^{\frac{1}{q'}}$$

$$\left. + \left(\fint_{Q(R)} |g_R - g(x)|^{q'} \, dx \right)^{\frac{1}{q'}} \right] = I + II + III + IV.$$

We obtain, using the assumptions on A :

$$\int_{Q(R)} |Dw|^p \, dx \leq c \int_{Q(R)} H(Du) dx \left[\left(\fint_{Q(R)} \sigma(x - x_0, R)^{q'} \, dx \right)^{\frac{1}{q'}} \right.$$

$$+ \left(\fint_{Q(R)} \omega(|u_R - u|^2)^{q'} \, dx \right)^{\frac{1}{q'}} + \left(\fint_{Q(R)} \omega(|u_R - v|^2)^{q'} \, dx \right)^{\frac{1}{q'}}$$

$$\left. + \left(\fint_{Q(R)} |g_R - g(x)|^{q'} \, dx \right)^{\frac{1}{q'}} \right] = I + II + III + IV.$$

Let us use the above mentioned regularity theorem by Uhlenbeck in the first part of I, in II and III Hölder, Jensen and Poincaré inequality and in IV the assumption on g,

we have

$$\int_{Q(R)} |Du|^p \, dx \leq C \Bigg\{ \left(\frac{r}{R}\right)^{\lambda} + \left(⨍_{Q(R)} \sigma(x,R) dx \right)^{\frac{q-1}{q}} +$$

$$+\omega \left(R^{p-m} \int_{Q(R)} |Du|^p dx \right)^{\frac{q-1}{q}} + \eta(g,R) \Bigg\} \cdot \int_{Q(2R)} H(Du) dx.$$

we have

$$\int_{Q(R)} |Du|^p \, dx \le C \left\{ \left(\frac{r}{R}\right)^{\lambda} + \left(\fint_{Q(R)} \sigma(x,R) dx \right)^{\frac{q-1}{q}} + \right.$$

$$\left. + \omega \left(R^{p-m} \int_{Q(R)} |Du|^p dx \right)^{\frac{q-1}{q}} + \eta(g,R) \right\} \cdot \int_{Q(2R)} H(Du) dx.$$

Furthermore recalling the VMO assumption we have

$$\fint_{B(R)} \sigma(x,R) dx \to 0, \quad , \quad \eta(g,R) \to 0 \quad \text{as} \quad R \to 0.$$

we have

$$\int_{Q(R)} |Du|^p \, dx \le C\Bigg\{ \left(\frac{r}{R}\right)^{\lambda} + \left(\fint_{Q(R)} \sigma(x,R) dx \right)^{\frac{q-1}{q}} +$$

$$+\omega\left(R^{p-m} \int_{Q(R)} |Du|^p dx \right)^{\frac{q-1}{q}} + \eta(g,R) \Bigg\} \cdot \int_{Q(2R)} H(Du) dx.$$

Furthermore recalling the VMO assumption we have

$$\fint_{B(R)} \sigma(x,R) dx \to 0, \quad , \quad \eta(g,R) \to 0 \quad \text{as} \quad R \to 0.$$

We conclude using "A useful lemma" and another result contained in the book "Multiple integrals in the calculus of variations and nonlinear elliptic systems", Annals of Math. Studies Princeton Univ. Press 1983 by Giaquinta.

In recent years, there has been an increase in interest in problems related to $p(x)$-growth.

In recent years, there has been an increase in interest in problems related to $p(x)$-growth. They appear not only in purely mathematical context but also in some physical problems.

In recent years, there has been an increase in interest in problems related to $p(x)$-growth. They appear not only in purely mathematical context but also in some physical problems. Zhikov in 1997 treated a $p(x)$-growth functional related to the thermistor problem, and got some results on the higher integrability of the gradient of minimizers.

In recent years, there has been an increase in interest in problems related to $p(x)$-growth. They appear not only in purely mathematical context but also in some physical problems.
Zhikov in 1997 treated a $p(x)$-growth functional related to the thermistor problem, and got some results on the higher integrability of the gradient of minimizers.
Rajagopal and Ružička in 2001 proposed a model of electrorheological fluid, using equations with $p(x)$-growth term.

In recent years, there has been an increase in interest in problems related to $p(x)$-growth. They appear not only in purely mathematical context but also in some physical problems.
Zhikov in 1997 treated a $p(x)$-growth functional related to the thermistor problem, and got some results on the higher integrability of the gradient of minimizers.
Rajagopal and Ružička in 2001 proposed a model of electrorheological fluid, using equations with $p(x)$-growth term.
Moreover, Acerbi and Mingione in 2002 obtained some regularity results for stationary electrorheological fluid.

In recent years, there has been an increase in interest in problems related to $p(x)$-growth. They appear not only in purely mathematical context but also in some physical problems. Zhikov in 1997 treated a $p(x)$-growth functional related to the thermistor problem, and got some results on the higher integrability of the gradient of minimizers.
Rajagopal and Ružička in 2001 proposed a model of electrorheological fluid, using equations with $p(x)$-growth term. Moreover, Acerbi and Mingione in 2002 obtained some regularity results for stationary electrorheological fluid. Let us now consider

$$\mathcal{F}(u) = \int_\Omega (g^{\alpha\beta}(x) h_{ij}(u) D_\alpha u^i D_\beta u^j)^{p(x)/2} dx,$$

where $g^{\alpha\beta}(x)$, $h_{ij}(u)$ and $p(x)$ satisfies

where $g^{\alpha\beta}(x)$, $h_{ij}(u)$ and $p(x)$ satisfies

(C1) **There exist positive constants** $\lambda_g, \Lambda_g, \lambda_h, \Lambda_h$:

$$\lambda_g|\zeta|^2 \leq g^{\alpha\beta}(x)\zeta_\alpha\zeta_\beta \leq \Lambda_g|\zeta|^2, \quad \lambda_h|\eta|^2 \leq h_{ij}(u)\eta^i\eta^j \leq \Lambda_h|\eta|^2$$

for all $x \in \Omega$, $\zeta \in \mathbb{R}^m$ and $u, \eta \in \mathbb{R}^n$.

where $g^{\alpha\beta}(x)$, $h_{ij}(u)$ and $p(x)$ satisfies

(C1) **There exist positive constants** $\lambda_g, \Lambda_g, \lambda_h, \Lambda_h$:

$$\lambda_g|\zeta|^2 \leq g^{\alpha\beta}(x)\zeta_\alpha\zeta_\beta \leq \Lambda_g|\zeta|^2, \quad \lambda_h|\eta|^2 \leq h_{ij}(u)\eta^i\eta^j \leq \Lambda_h|\eta|^2$$

for all $x \in \Omega$, $\zeta \in \mathbb{R}^m$ and $u, \eta \in \mathbb{R}^n$.

(C2) **There exist constants** $\exists\, 0 < \tau, \tau' < 1$, $0 < \sigma \leq 1$, L_p, L_g, L_h:

where $g^{\alpha\beta}(x)$, $h_{ij}(u)$ and $p(x)$ satisfies

(C1) **There exist positive constants** $\lambda_g, \Lambda_g, \lambda_h, \Lambda_h$:

$$\lambda_g|\zeta|^2 \leq g^{\alpha\beta}(x)\zeta_\alpha\zeta_\beta \leq \Lambda_g|\zeta|^2, \quad \lambda_h|\eta|^2 \leq h_{ij}(u)\eta^i\eta^j \leq \Lambda_h|\eta|^2$$

for all $x \in \Omega$, $\zeta \in \mathbb{R}^m$ and $u, \eta \in \mathbb{R}^n$.

(C2) **There exist constants** $\exists\ 0 < \tau, \tau' < 1$, $0 < \sigma \leq 1$, L_p, L_g, L_h:

$$|p(x) - p(y)| \leq L_p|x - y|^\sigma/2 =: \omega_p(|x - y|/2), \forall x, y \in \Omega,$$

$$|g^{\alpha\beta}(x) - g^{\alpha\beta}(y)| \leq L_g|x - y|^\tau =: \omega_g(|x - y|), \forall x, y \in \Omega,$$

$$|h_{ij}(u) - h_{ij}(v)| \leq L_h|u - v|^{\tau'} =: \omega_h(|u - v|^2), \forall u, v \in \mathbb{R}^n.$$

where $g^{\alpha\beta}(x)$, $h_{ij}(u)$ and $p(x)$ satisfies

(C1) **There exist positive constants** $\lambda_g, \Lambda_g, \lambda_h, \Lambda_h$:

$$\lambda_g|\zeta|^2 \leq g^{\alpha\beta}(x)\zeta_\alpha\zeta_\beta \leq \Lambda_g|\zeta|^2, \quad \lambda_h|\eta|^2 \leq h_{ij}(u)\eta^i\eta^j \leq \Lambda_h|\eta|^2$$

for all $x \in \Omega$, $\zeta \in \mathbb{R}^m$ and $u, \eta \in \mathbb{R}^n$.

(C2) **There exist constants** $\exists\, 0 < \tau, \tau' < 1$, $0 < \sigma \leq 1$, L_p, L_g, L_h:

$$|p(x) - p(y)| \leq L_p|x - y|^\sigma/2 =: \omega_p(|x - y|/2), \forall x, y \in \Omega,$$
$$|g^{\alpha\beta}(x) - g^{\alpha\beta}(y)| \leq L_g|x - y|^\tau =: \omega_g(|x - y|), \forall x, y \in \Omega,$$
$$|h_{ij}(u) - h_{ij}(v)| \leq L_h|u - v|^{\tau'} =: \omega_h(|u - v|^2), \forall u, v \in \mathbb{R}^n.$$

(C3) $$2 \leq \gamma_1 := \inf_{x\in\Omega} p(x) \leq \sup_{x\in\Omega} p(x) =: \gamma_2 < +\infty.$$

The study in cooperation with A. Tachikawa is continued obtaining,
- [RTT T.A.M.S., 2013]

The study in cooperation with A. Tachikawa is continued obtaining,
- [RTT T.A.M.S., 2013]
- [RT Nonlinear Analysis, 2013]

The study in cooperation with A. Tachikawa is continued obtaining,

- [RTT T.A.M.S., 2013]
- [RT Nonlinear Analysis, 2013]
- [RT Annales Institute Henry Poincaré - Analyse non lin., 2016].

- [RTT, 2013] if in this paper $g^{\alpha\beta}(x)$ are in the class VMO instead of assumption on g in (C2),
$\Rightarrow$ Then, $u \in C^{0,\alpha}(\Omega_0)$ for some $\alpha \in (0,1)$, where Ω_0 is an open subset of Ω with $\mathcal{H}^{m-\gamma_1}(\Omega \setminus \Omega_0) = 0$.

- [RTT, 2013] if in this paper $g^{\alpha\beta}(x)$ are in the class VMO instead of assumption on g in (C2),
$\Rightarrow$ Then, $u \in C^{0,\alpha}(\Omega_0)$ for some $\alpha \in (0,1)$, where Ω_0 is an open subset of Ω with $\mathcal{H}^{m-\gamma_1}(\Omega \setminus \Omega_0) = 0$.
Moreover, if $g^{\alpha\beta}(x)$ and $h_{ij}(u)$ are Hölder continuous,
$u \in C^{1,\alpha'}(\Omega_0)$ for some $\alpha' \in (0,1)$.

- [RTT, 2013] if in this paper $g^{\alpha\beta}(x)$ are in the class VMO instead of assumption on g in (C2),
$\Rightarrow$ Then, $u \in C^{0,\alpha}(\Omega_0)$ for some $\alpha \in (0,1)$, where Ω_0 is an open subset of Ω with $\mathcal{H}^{m-\gamma_1}(\Omega \setminus \Omega_0) = 0$.
Moreover, if $g^{\alpha\beta}(x)$ and $h_{ij}(u)$ are Hölder continuous, $u \in C^{1,\alpha'}(\Omega_0)$ for some $\alpha' \in (0,1)$.
-[RT, Nonlinear Analysis 2013] if in this paper we delete σ

- [RTT, 2013] if in this paper $g^{\alpha\beta}(x)$ are in the class VMO instead of assumption on g in (C2),
$\Rightarrow$ Then, $u \in C^{0,\alpha}(\Omega_0)$ for some $\alpha \in (0,1)$, where Ω_0 is an open subset of Ω with $\mathcal{H}^{m-\gamma_1}(\Omega \setminus \Omega_0) = 0$.
Moreover, if $g^{\alpha\beta}(x)$ and $h_{ij}(u)$ are Hölder continuous, $u \in C^{1,\alpha'}(\Omega_0)$ for some $\alpha' \in (0,1)$.
-[RT, Nonlinear Analysis 2013] if in this paper we delete σ and add "one-sided condition"

$$\exists \lambda_h^* > 0 : -\frac{1}{2} u^k h_{ij,k}(u) \eta^i \eta^j \leq \lambda_h^* |\eta|^2$$

for all $u \in B_M(0)$ and $\eta \in \mathbb{R}^n$, where $h_{ij,k} = \partial h_{ij} \partial u^k$,

- [RTT, 2013] if in this paper $g^{\alpha\beta}(x)$ are in the class VMO instead of assumption on g in (C2),
$\Rightarrow$ Then, $u \in C^{0,\alpha}(\Omega_0)$ for some $\alpha \in (0,1)$, where Ω_0 is an open subset of Ω with $\mathcal{H}^{m-\gamma_1}(\Omega \setminus \Omega_0) = 0$.
Moreover, if $g^{\alpha\beta}(x)$ and $h_{ij}(u)$ are Hölder continuous, $u \in C^{1,\alpha'}(\Omega_0)$ for some $\alpha' \in (0,1)$.
-[RT, Nonlinear Analysis 2013] if in this paper we delete σ and add "one-sided condition"

$$\exists \lambda_h^* > 0 : -\frac{1}{2} u^k h_{ij,k}(u) \eta^i \eta^j \leq \lambda_h^* |\eta|^2$$

for all $u \in B_M(0)$ and $\eta \in \mathbb{R}^n$, where $h_{ij,k} = \partial h_{ij} \partial u^k$,
$\Rightarrow$ Then, $u \in C^{1,\alpha}(\Omega)$, for some $\alpha \in (0,1)$.

- [RTT, 2013] if in this paper $g^{\alpha\beta}(x)$ are in the class VMO instead of assumption on g in (C2),
$\Rightarrow$ Then, $u \in C^{0,\alpha}(\Omega_0)$ for some $\alpha \in (0,1)$, where Ω_0 is an open subset of Ω with $\mathcal{H}^{m-\gamma_1}(\Omega \setminus \Omega_0) = 0$.
Moreover, if $g^{\alpha\beta}(x)$ and $h_{ij}(u)$ are Hölder continuous,
$u \in C^{1,\alpha'}(\Omega_0)$ for some $\alpha' \in (0,1)$.
-[RT, Nonlinear Analysis 2013] if in this paper we delete σ and add "one-sided condition"

$$\exists \lambda_h^* > 0 : -\frac{1}{2} u^k h_{ij,k}(u) \eta^i \eta^j \leq \lambda_h^* |\eta|^2$$

for all $u \in B_M(0)$ and $\eta \in \mathbb{R}^n$, where $h_{ij,k} = \partial h_{ij} \partial u^k$,
$\Rightarrow$ Then, $u \in C^{1,\alpha}(\Omega)$, for some $\alpha \in (0,1)$.
-[RT, AIHP, 2016]

- [RTT, 2013] if in this paper $g^{\alpha\beta}(x)$ are in the class VMO instead of assumption on g in (C2),
$\Rightarrow$ Then, $u \in C^{0,\alpha}(\Omega_0)$ for some $\alpha \in (0,1)$, where Ω_0 is an open subset of Ω with $\mathcal{H}^{m-\gamma_1}(\Omega \setminus \Omega_0) = 0$.
Moreover, if $g^{\alpha\beta}(x)$ and $h_{ij}(u)$ are Hölder continuous,
$u \in C^{1,\alpha'}(\Omega_0)$ for some $\alpha' \in (0,1)$.
-[RT, Nonlinear Analysis 2013] if in this paper we delete σ and add "one-sided condition"

$$\exists \lambda_h^* > 0 : -\frac{1}{2} u^k h_{ij,k}(u)\eta^i\eta^j \leq \lambda_h^* |\eta|^2$$

for all $u \in B_M(0)$ and $\eta \in \mathbb{R}^n$, where $h_{ij,k} = \partial h_{ij} \partial u^k$,
$\Rightarrow$ Then, $u \in C^{1,\alpha}(\Omega)$, for some $\alpha \in (0,1)$.
-[RT, AIHP, 2016]
$\Rightarrow$ Then, u is Hölder continuous near the boundary $\partial\Omega$.

For locally bounded minimizers, the $C^{1,\alpha}$ result contained in [RTT, T.A.M.S., 2013] has been improved as

$$\dim_{\mathcal{H}}(\Omega \setminus \Omega_0) < m - [\gamma_1] - 1,$$

where [ ] stands for the Gauss symbol, by
A. Tachikawa in Calculus of Variations and P.D.E., 2014.

In [RTT T.A.M.S., 2013] the technique to obtain higher integrability properties for local minimizers employs a direct approach as well as classical inequalities by Jensen and Poincaré.

In [RTT T.A.M.S., 2013] the technique to obtain higher integrability properties for local minimizers employs a direct approach as well as classical inequalities by Jensen and Poincaré.
In [RT Nonlinear Analysis, 2013] and [RT AIHP, 2016] we gain the results by contradiction.