Characterizations for the Riesz potential and its commutators on Orlicz and generalized Orlicz-Morrey spaces

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Let $0 < \alpha < n$. The fractional maximal operator M_α and the Riesz potential operator I_α are defined by

Fractional maximal operator

$$M_\alpha f(x) = \sup_{t>0} |B(x,t)|^{-1+\frac{\alpha}{n}} \int_{B(x,t)} |f(y)| dy,$$

Riesz potential

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy,$$

respectively. Here and everywhere in the sequel $B(x,r)$ is the ball in $\mathbb{R}^n$ of radius r centered at x and $|B(x,r)| = v_n r^n$ is its Lebesgue measure, where v_n is the volume of the unit ball in $\mathbb{R}^n$.

Remark

If $\alpha = 0$, then $M \equiv M_0$ is the well known Hardy-Littlewood maximal operator.

Remark

Recall that, for $0 < \alpha < n$,

$$M_\alpha f(x) \leq v_n^{\frac{\alpha}{n}-1} I_\alpha(|f|)(x).$$

The commutators generated by a suitable function b and the operators M_α and I_α are formally defined by

Commutators of fractional maximal operator

$$[b, M_\alpha]f = M_\alpha(bf) - bM_\alpha(f),$$

Commutators of Riesz potential

$$[b, I_\alpha]f = I_\alpha(bf) - bI_\alpha(f),$$

respectively.

Given a measurable function b the operators $M_{b,\alpha}$ and $|b, I_\alpha|$ are defined by

$$M_{b,\alpha}(f)(x) = \sup_{t>0} |B(x,t)|^{-1+\frac{\alpha}{n}} \int_{B(x,t)} |b(x) - b(y)||f(y)|dy$$

$$|b, I_\alpha| f(x) = \int_{\mathbb{R}^n} \frac{|b(x) - b(y)|}{|x-y|^{n-\alpha}} f(y)dy,$$

respectively. If $\alpha = 0$, then $M_{b,0} \equiv M_b$ is the sublinear commutator of the Hardy-Littlewood maximal operator. Recall that, for $0 < \alpha < n$,

Remark

$$M_{b,\alpha}(f)(x) \le v_n^{\frac{\alpha}{n}-1} |b, I_\alpha|(|f|)(x)$$

Remark

$$|[b, I_\alpha]f(x)| \le |b, I_\alpha|(|f|)(x).$$

For a function b defined on $\mathbb{R}^n$, we denote

$$b^-(x) = \begin{cases} 0\ , & \text{if } b(x) \geq 0 \\ |b(x)|, & \text{if } b(x) < 0 \end{cases}$$

and $b^+(x) = |b(x)| - b^-(x)$. Obviously, $b^+(x) - b^-(x) = b(x)$. The following relations between $[b, M_\alpha]$ and $M_{b,\alpha}$ are valid :

Lemma

Let $0 \leq \alpha < n$ and b be any non-negative locally integrable function. Then

$$|[b, M_\alpha]f(x)| \leq M_{b,\alpha}(f)(x), \qquad x \in \mathbb{R}^n$$

holds for all $f \in L^1_{\rm loc}(\mathbb{R}^n)$.

Lemma

Let $0 \leq \alpha < n$. If b is any locally integrable function on $\mathbb{R}^n$, then

$$|[b, M_\alpha]f(x)| \leq M_{b,\alpha}(f)(x) + 2b^-(x)M_\alpha f(x), \qquad x \in \mathbb{R}^n \tag{1}$$

holds for all $f \in L^1_{\rm loc}(\mathbb{R}^n)$.

Boundedness of the classical operators of harmonic analysis, such as maximal operator M, fractional maximal operator M_α and Riesz potential operator I_α have been extensively investigated in various function spaces. Results on weak and strong type inequalities for operators of this kind in Lebesgue spaces are classical and can be found for example in [BenSh, St70, St93, Torch].

- C. Bennett and R. Sharpley, *Interpolation of operators*, Academic Press, Boston, 1988.
- E.M. Stein, *Singular integrals and differentiability of functions*, Princeton University Press, Princeton, NJ, 1970.
- E.M. Stein, *Harmonic Analysis: Real Variable Methods, Orthogonality and Oscillatory Integrals*, Princeton Univ. Press, Princeton NJ, 1993.
- A. Torchinsky, Real Variable Methods in Harmonic Analysis, Pure and Applied Math. 123, Academic Press, New York, 1986.

Generalizations of these results to Zygmund spaces are presented in [BenSh]. An exhaustive treatment of the problem of boundedness of such operators in Lorentz and Lorentz-Zygmund spaces is given in [BenRudn].

C. Bennett, K. Rudnick, *On Lorentz-Zygmund spaces*, Dissertationes Math. 175 (1980).

See also [EdmundsGurOp1], [EdmundsGurOp2] for further extensions in the framework of generalized Lorentz-Zygmund spaces.

D.E. Edmunds, P. Gurka, B. Opic, *Double exponential integrability of convolution operators in generalized Lorentz-Zygmund spaces*, Indiana Univ. Math. J. 44 (1995).

D.E. Edmunds, P. Gurka, B. Opic, *On embeddings of logarithmic Bessel potential spaces*, J. Funct. Anal. 146 (1997) 116-150.

As far as Orlicz spaces are concerned, a characterization of Young functions Φ having the property that the operator M is of weak or strong type from the Orlicz space L^{Φ} into itself is known (see for example [Kita1], [KokKrbec]).

H. Kita, *On maximal functions in Orlicz spaces*, Proc. Amer. Math. Soc. 124 (1996) 3019-3025.

V. Kokilashvili, M. M. Krbec, *Weighted Inequalities in Lorentz and Orlicz Spaces*, World Scientific, Singapore, 1991.

In [O'Neil], [Torch1] conditions on Young functions Φ and Ψ are given for the operator I_α to be bounded from L^{Φ} into L^{Ψ} under some restrictions involving the growths and certain monotonicity properties of Φ and Ψ.

R. O'Neil, *Fractional integration in Orlicz spaces*, Trans. Amer. Math. Soc. 115 (1965) 300-328.

A. Torchinsky, *Interpolation of operators and Orlicz classes*, Studia Math. 59 (1976) 177-207.

In [Cianchi, 1999], Cianchi found necessary and sufficient conditions on general Young functions Φ and Ψ ensuring that the operators M_α and I_α are of weak or strong type from L^Φ into L^Ψ.

A. Cianchi, *Strong and weak type inequalities for some classical operators in Orlicz spaces*, J. London Math. Soc. (2) 60 (1) (1999) 247-286.

Another boundedness statement with only sufficient conditions for the operator I_α on Orlicz spaces was given by Nakai [Nakai]. Note that in [Nakai] a more general case of generalized fractional integrals was studied.

E. Nakai, *On generalized fractional integrals*, Taiwanese J. Math. 5 (2001) 587-602.

Commutators of classical operators of harmonic analysis play an important role in various topics of analysis and PDE, where in particular in [Chanillo] it was shown that the commutator $[b, I_\alpha]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for $1 < p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and $b \in BMO(\mathbb{R}^n)$.

S. Chanillo, *A note on commutators*, Indiana Univ. Math. J. 31 (1) (1982) 7-16.

In [FuYangYuan], Fu et al. gave the sufficient conditions for the boundedness of the commutator $[b, I_\alpha]$ on Orlicz spaces.

X. Fu, D. Yang, W. Yuan, *Generalized fractional integrals and their commutators over non-homogeneous metric measure spaces*, Taiwanese J. Math. 18 (2) (2014) 509-557.

The following theorem is valid.

Theorem ([Chanillo], [DGSDebr2017])

Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Then $M_{b,\alpha}$, $[b, I_\alpha]$ and $|b, I_\alpha|$ are bounded operators from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ if and only if $b \in BMO(\mathbb{R}^n)$.

The proof of Theorem for $[b, I_\alpha]$ was given in [Chanillo] and for $M_{b,\alpha}$ and $|b, I_\alpha|$ were given in [DGSDebr2017].

F. Deringoz, V. Guliyev, S. Samko, *Vanishing generalized Orlicz-Morrey spaces and fractional maximal operator*, Publ. Math. Debrecen. 90 (1-2) (2017), 125-147.

The purpose of the talk

The main purpose of this talk is to give characterizations for the strong/weak boundedness of the fractional maximal and Riesz potential operators on Orlicz spaces. Our characterizations for the boundedness of the operators M_α and I_α are different from the ones in [Cianchi, 1999].

Moreover, as an application of these results we consider the boundedness of the commutators of M_α and I_α on Orlicz spaces when b belongs to the BMO and Lipschitz spaces, respectively.

Orlicz space was first introduced by Orlicz as a generalizations of Lebesgue spaces L^p. Since then this space has been one of important functional frames in the mathematical analysis, and especially in real and harmonic analysis. Orlicz space is also an appropriate substitute for L^1 space when L^1 space does not work.

First, we recall the definition of Young functions.

A

function $\Phi : [0, \infty) \to [0, \infty]$ is called a Young function if Φ is convex, left-continuous, $\lim_{r\to+0} \Phi(r) = \Phi(0) = 0$ and $\lim_{r\to\infty} \Phi(r) = \infty$.

From the convexity and $\Phi(0) = 0$ it follows that any Young function is increasing. If there exists $s \in (0, \infty)$ such that $\Phi(s) = \infty$, then $\Phi(r) = \infty$ for $r \geq s$.

The class $\mathcal{Y}$

The set of Young functions such that

$$0 < \Phi(r) < \infty \qquad \text{for} \qquad 0 < r < \infty$$

will be denoted by $\mathcal{Y}$. If $\Phi \in \mathcal{Y}$, then Φ is absolutely continuous on every closed interval in $[0, \infty)$ and bijective from $[0, \infty)$ to itself.

Generalized inverse

For a Young function Φ and $0 \leq s \leq \infty$, let

$$\Phi^{-1}(s) = \inf\{r \geq 0 : \Phi(r) > s\}.$$

If $\Phi \in \mathcal{Y}$, then Φ^{-1} is the usual inverse function of Φ.

It is well known that

Lemma

$$r \le \Phi^{-1}(r)\widetilde{\Phi}^{-1}(r) \le 2r \qquad \text{for } r \ge 0,$$

where $\widetilde{\Phi}(r)$ is defined by

Young conjugate

$$\widetilde{\Phi}(r) = \begin{cases} \sup\{rs - \Phi(s) : s \in [0,\infty)\} & , \quad r \in [0,\infty) \\ \infty & , \quad r = \infty. \end{cases}$$

A Young function Φ is said to satisfy the Δ_2-condition, denoted also as $\Phi \in \Delta_2$, if

Δ_2-condition

$$\Phi(2r) \leq C\Phi(r), \qquad r > 0$$

for some $C > 1$. If $\Phi \in \Delta_2$, then $\Phi \in \mathcal{Y}$. A Young function Φ is said to satisfy the ∇_2-condition, denoted also by $\Phi \in \nabla_2$, if

∇_2-condition

$$\Phi(r) \leq \frac{1}{2C}\Phi(Cr), \qquad r \geq 0$$

for some $C > 1$.

Orlicz space

For a Young function Φ, the set

$$L^{\Phi}(\mathbb{R}^n) = \left\{ f \in L^1_{\rm loc}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \Phi(k|f(x)|)dx < \infty \text{ for some } k > 0 \right\}$$

is called Orlicz space.

$L^{\Phi}(\mathbb{R}^n)$ is a Banach space with respect to the norm

Luxemburg Norm

$$\|f\|_{L^\Phi} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \Phi\Big(\frac{|f(x)|}{\lambda}\Big) dx \le 1 \right\}.$$

If $\Phi(r) = r^p$, $1 \le p < \infty$, then $L^{\Phi}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. If $\Phi(r) = 0$, $(0 \le r \le 1)$ and $\Phi(r) = \infty$, $(r > 1)$, then $L^{\Phi}(\mathbb{R}^n) = L^{\infty}(\mathbb{R}^n)$. The space $L^{\Phi}_{\rm loc}(\mathbb{R}^n)$ is defined as the set of all functions f such that $f\chi_B \in L^{\Phi}(\mathbb{R}^n)$ for all balls $B \subset \mathbb{R}^n$.

Weak Orlicz space

Let Φ be a Young function. The weak Orlicz space is defined as

$$WL^{\Phi}(\mathbb{R}^n) := \{f \in L^1_{\rm loc}(\mathbb{R}^n) : \|f\|_{WL^\Phi} < \infty\},$$

where

$$\|f\|_{WL^\Phi} = \inf\left\{\lambda > 0 \ : \ \sup_{t>0} \Phi\left(\frac{t}{\lambda}\right) d_f(t) \ \leq 1\right\},$$

and $d_f(t) = |\{x \in \mathbb{R}^n : |f(x)| > t\}|$.

Ex 1. Let either $1 \leq p < \infty$, and $\Phi(t) = t^p$, or $p = \infty$ and $\Phi(t) = \infty\chi_{(1,\infty)}(t)$.
Then

$$L^\Phi(\mathbb{R}^n) = L^p(\mathbb{R}^n),$$

the Lebesgue space.
Ex 2. $\Phi(t) \approx t^p \log(1+t)^\alpha$, either $p > 1$ and $\alpha \in \mathbb{R}$, or $p = 1$ and $\alpha \geq 0$.
Then

$$L^\Phi(\mathbb{R}^n) = L^p(LogL)^\alpha(\mathbb{R}^n),$$

the Zygmund space.
Ex 3. $\Phi(t) \approx e^{t^\beta} - 1$, $\beta > 0$.
Then

$$L^\Phi(\mathbb{R}^n) = expL^\beta(\mathbb{R}^n).$$

We recall that, for functions Φ and Ψ from $[0,\infty)$ into $[0,\infty]$, the function Ψ is said to dominate Φ globally if there exists a positive constant c such that $\Phi(s) \le \Psi(cs)$ for all $s \ge 0$.
In the theorems below we also use the notation

$$\widetilde{\Psi_P}(s) = \int_0^s r^{P'-1}(\mathcal{B}_P^{-1}(r^{P'}))^{P'}\,dr, \tag{2}$$

where $1 < P \le \infty$ and $\widetilde{\Psi_P}(s)$ is the Young conjugate function to $\Psi_P(s)$, and

$$\Phi_P(s) = \int_0^s r^{P'-1}(\mathcal{A}_P^{-1}(r^{P'}))^{P'}\,dr, \tag{3}$$

where $\mathcal{B}_P^{-1}(s)$ and $\mathcal{A}_P^{-1}(s)$ are inverses to

$$\mathcal{B}_P(s) = \int_0^s \frac{\Psi(t)}{t^{1+P'}}\,dt \quad \text{and} \quad \mathcal{A}_P(s) = \int_0^s \frac{\widetilde{\Phi}(t)}{t^{1+P'}}\,dt,$$

respectively. These functions $\Psi_P(s)$ and $\Phi_P(s)$ are used below with $P = \frac{n}{\alpha}$.

Cianchi (1999) found the necessary and sufficient conditions for the boundedness of I_α on Orlicz spaces.

Cianchi's Theorem

Let $0<\alpha<n$. Let Φ and Ψ Young functions and let $\Phi_{n/\alpha}$ and $\Psi_{n/\alpha}$ be the Young functions defined as in (3) and (2), respectively. Then
(i) I_α is bounded from $L^\Phi(\mathbb{R}^n)$ to $WL^\Psi(\mathbb{R}^n)$ if and only if

$$\int_0^1 \widetilde{\Phi}(t)/t^{1+n/(n-\alpha)}dt<\infty \text{ and } \Phi_{n/\alpha} \text{ dominates } \Psi \text{ globally.} \tag{4}$$

(ii) I_α is bounded from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$ if and only if

$$\int_0^1 \widetilde{\Phi}(t)/t^{1+n/(n-\alpha)}dt<\infty,\ \int_0^1 \Psi(t)/t^{1+n/(n-\alpha)}dt<\infty, \\ \Phi \text{ dominates } \Psi_{n/\alpha} \text{ globally and } \Phi_{n/\alpha} \text{ dominates } \Psi \text{ globally.} \tag{5}$$

The following theorem gives necessary and sufficient conditions for the boundedness of the operator I_α from $WL^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$ and from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

Main Result

Let $0 < \alpha < n$ and $\Phi, \Psi \in \mathcal{Y}$.
1. The condition

$$r^\alpha \Phi^{-1}\big(r^{-n}\big) + \int_r^\infty \Phi^{-1}\big(t^{-n}\big) t^\alpha \frac{dt}{t} \lesssim \Psi^{-1}\big(r^{-n}\big) \tag{6}$$

is sufficient for the boundedness of I_α from $L^\Phi(\mathbb{R}^n)$ to $WL^\Psi(\mathbb{R}^n)$. Moreover, if $\Phi \in \nabla_2$, the condition (6) is sufficient for the boundedness of I_α from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.
2. The condition $r^\alpha \Phi^{-1}\big(r^{-n}\big) \lesssim \Psi^{-1}\big(r^{-n}\big)$ is necessary for the boundedness of I_α from $L^\Phi(\mathbb{R}^n)$ to $WL^\Psi(\mathbb{R}^n)$ and from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

Main Result

3. If the regularity condition

$$\int_r^\infty \Phi^{-1}\big(t^{-n}\big)t^\alpha \frac{dt}{t} \lesssim r^\alpha \Phi^{-1}\big(r^{-n}\big) \tag{7}$$

holds, then the condition $r^\alpha \Phi^{-1}\big(r^{-n}\big) \lesssim \Psi^{-1}\big(r^{-n}\big)$ is necessary and sufficient for the boundedness of I_α from $L^\Phi(\mathbb{R}^n)$ to $WL^\Psi(\mathbb{R}^n)$. Moreover, if $\Phi \in \nabla_2$, the condition $r^\alpha \Phi^{-1}\big(r^{-n}\big) \lesssim \Psi^{-1}\big(r^{-n}\big)$ is necessary and sufficient for the boundedness of I_α from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

V.S. Guliyev, F. Deringoz, S.G. Hasanov, *Riesz potential and its commutators on Orlicz spaces*, J. Inequal. Appl. 2017, 2017:75.

Remark

Let $0 < \alpha < n$, $\Phi, \Psi \in \mathcal{Y}$ and the regularity condition (7) holds, then:
1) Condition (4) holds if and only if condition $r^{\alpha}\Phi^{-1}\big(r^{-n}\big) \lesssim \Psi^{-1}\big(r^{-n}\big)$ holds.
2) Moreover if $\Phi \in \nabla_2$, then condition (5) holds if and only if $r^{\alpha}\Phi^{-1}\big(r^{-n}\big) \lesssim \Psi^{-1}\big(r^{-n}\big)$ holds.

The following result is due to Nakai.

Nakai's Result

Let $0 < \alpha < n$ and $\Phi, \Psi \in \mathcal{Y}$. Assume that the conditions $r^{\alpha}\Phi^{-1}\big(r^{-n}\big) \lesssim \Psi^{-1}\big(r^{-n}\big)$ and

$$\int_r^{\infty} \Phi^{-1}\big(t^{-n}\big)t^{\alpha}\frac{dt}{t} \lesssim r^{\alpha}\Phi^{-1}\big(r^{-n}\big)$$

hold. Then the operator I_α is bounded from $L^{\Phi}(\mathbb{R}^n)$ to $WL^{\Psi}(\mathbb{R}^n)$. Moreover, if $\Phi \in \nabla_2$, then I_α is bounded from $L^{\Phi}(\mathbb{R}^n)$ to $L^{\Psi}(\mathbb{R}^n)$.

E. Nakai, *On generalized fractional integrals*, Taiwanese J. Math. 5 (2001) 587-602.

The following theorem gives necessary and sufficient conditions for the boundedness of the operator M_α from $L^\Phi(\mathbb{R}^n)$ to $WL^\Psi(\mathbb{R}^n)$ and from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

Main Result

Let $0 < \alpha < n$, Φ, Ψ be Young functions and $\Phi \in \mathcal{Y}$. The condition

$$r^{-\frac{\alpha}{n}} \Phi^{-1}\big(r\big) \le C \Psi^{-1}\big(r\big) \qquad (8)$$

for all $r > 0$, where $C > 0$ does not depend on r, is necessary and sufficient for the boundedness of M_α from $L^\Phi(\mathbb{R}^n)$ to $WL^\Psi(\mathbb{R}^n)$. Moreover, if $\Phi \in \nabla_2$, the condition (8) is necessary and sufficient for the boundedness of M_α from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

V.S. Guliyev, F. Deringoz, S.G. Hasanov, *Fractional maximal function and its commutators on Orlicz spaces*, submitted.

In [FuYangYuan], Fu et al. found the sufficient conditions for the boundedness of the commutator $[b, I_\alpha]$ on Orlicz spaces as follows.

Theorem

Let $0 < \alpha < n$ and $b \in BMO(\mathbb{R}^n)$. Let Φ be a Young function and Ψ defined, via its inverse, by setting, for all $t \in (0, \infty)$, $\Psi^{-1}(t) := \Phi^{-1}(t)t^{-\alpha/n}$. If $\Phi, \Psi \in \Delta_2 \cap \nabla_2$, then $[b, I_\alpha]$ is bounded from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

X. Fu, D. Yang, W. Yuan, *Generalized fractional integrals and their commutators over non-homogeneous metric measure spaces*, Taiwanese J. Math. 18 (2) (2014) 509-557.

The following theorem gives necessary and sufficient conditions for the boundedness of the operator $|b, I_\alpha|$ from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

Main Result

Let $0 < \alpha < n$, $b \in BMO(\mathbb{R}^n)$ and $\Phi, \Psi \in \mathcal{Y}$.
1. If $\Phi \in \nabla_2$ and $\Psi \in \Delta_2$, then the condition

$$r^\alpha \Phi^{-1}\big(r^{-n}\big) + \int_r^\infty \Big(1 + \ln \frac{t}{r}\Big) \Phi^{-1}\big(t^{-n}\big) t^\alpha \frac{dt}{t} \lesssim \Psi^{-1}\big(r^{-n}\big) \tag{9}$$

is sufficient for the boundedness of $[b, I_\alpha]$ from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.
2. If $\Psi \in \Delta_2$, then the condition $r^\alpha \Phi^{-1}\big(r^{-n}\big) \lesssim \Psi^{-1}\big(r^{-n}\big)$ is necessary for the boundedness of $|b, I_\alpha|$ from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

Main Result

3. Let $\Phi \in \nabla_2$ and $\Psi \in \Delta_2$. If the condition

$$\int_r^\infty \Big(1 + \ln \frac{t}{r}\Big) \Phi^{-1}\big(t^{-n}\big) t^\alpha \frac{dt}{t} \lesssim r^\alpha \Phi^{-1}\big(r^{-n}\big) \tag{10}$$

holds, then the condition $r^\alpha \Phi^{-1}\big(r^{-n}\big) \lesssim \Psi^{-1}\big(r^{-n}\big)$ is necessary and sufficient for the boundedness of $|b, I_\alpha|$ from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

V.S. Guliyev, F. Deringoz, S.G. Hasanov, *Riesz potential and its commutators on Orlicz spaces*, J. Inequal. Appl. 2017, 2017:75.

The following theorem is also valid.

Theorem

Let $0 < \alpha < n$, $b \in L^1_{\rm loc}(\mathbb{R}^n)$ and $\Phi, \Psi \in \mathcal{Y}$.
1. If $\Phi \in \nabla_2$, $\Psi \in \Delta_2$ and the condition (9) holds, then the condition $b \in BMO(\mathbb{R}^n)$ is sufficient for the boundedness of $[b, I_\alpha]$ from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.
2. If $\Psi^{-1}(t) \lesssim \Phi^{-1}(t)t^{-\alpha/n}$, then the condition $b \in BMO(\mathbb{R}^n)$ is necessary for the boundedness of $|b, I_\alpha|$ from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.
3. If $\Phi \in \nabla_2$, $\Psi \in \Delta_2$, $\Psi^{-1}(t) \approx \Phi^{-1}(t)t^{-\alpha/n}$ and the condition

$$\int_r^\infty \Big(1 + \ln \frac{t}{r}\Big) \Phi^{-1}\big(t^{-n}\big) t^\alpha \frac{dt}{t} \lesssim r^\alpha \Phi^{-1}\big(r^{-n}\big)$$

holds, then the condition $b \in BMO(\mathbb{R}^n)$ is necessary and sufficient for the boundedness of $|b, I_\alpha|$ from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

V.S. Guliyev, F. Deringoz, S.G. Hasanov, *Riesz potential and its commutators on Orlicz spaces*, J. Inequal. Appl. 2017, 2017:75.

Main Result

Let $0 \leq \alpha < n$, $b \in BMO(\mathbb{R}^n)$ and $\Phi, \Psi \in \mathcal{Y}$.

1. If $\Phi \in \nabla_2$ and $\Psi \in \Delta_2$, then the condition

$$r^{\alpha}\Phi^{-1}\big(r^{-n}\big) + \sup_{r<t<\infty}\Big(1+\ln\frac{t}{r}\Big)\Phi^{-1}\big(t^{-n}\big)t^{\alpha} \lesssim \Psi^{-1}\big(r^{-n}\big) \tag{11}$$

is sufficient for the boundedness of $M_{b,\alpha}$ from $L^{\Phi}(\mathbb{R}^n)$ to $L^{\Psi}(\mathbb{R}^n)$.

2. If $\Psi \in \Delta_2$, then the condition $r^{\alpha}\Phi^{-1}\big(r^{-n}\big) \lesssim \Psi^{-1}\big(r^{-n}\big)$ is necessary for the boundedness of $M_{b,\alpha}$ from $L^{\Phi}(\mathbb{R}^n)$ to $L^{\Psi}(\mathbb{R}^n)$.

3. Let $\Phi \in \nabla_2$ and $\Psi \in \Delta_2$. If the condition

$$\sup_{r<t<\infty}\Big(1+\ln\frac{t}{r}\Big)\Phi^{-1}\big(t^{-n}\big)t^{\alpha} \lesssim r^{\alpha}\Phi^{-1}\big(r^{-n}\big) \tag{12}$$

holds, then the condition $r^{\alpha}\Phi^{-1}\big(r^{-n}\big) \lesssim \Psi^{-1}\big(r^{-n}\big)$ is necessary and sufficient for the boundedness of $M_{b,\alpha}$ from $L^{\Phi}(\mathbb{R}^n)$ to $L^{\Psi}(\mathbb{R}^n)$.

V.S. Guliyev, F. Deringoz, S.G. Hasanov, *Fractional maximal function and its commutators on Orlicz spaces*, submitted.

Corollary

Let $0 \le \alpha < n$, $b \in BMO(\mathbb{R}^n)$, $b^- \in L^\infty(\mathbb{R}^n)$ and Φ, Ψ be Young functions with $\Phi \in \nabla_2 \cap \mathcal{Y}$ and $\Psi \in \Delta_2$. Let also the condition (11) is satisfied. Then the operator $[b, M_\alpha]$ is bounded from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

The following theorem is also valid.

Theorem

Let $0 \leq \alpha < n$, $b \in L^1_{\rm loc}(\mathbb{R}^n)$ and $\Phi, \Psi \in \mathcal{Y}$.

1. If $\Phi \in \nabla_2$, $\Psi \in \Delta_2$ and the condition (11) holds, then the condition $b \in BMO(\mathbb{R}^n)$ is sufficient for the boundedness of $M_{b,\alpha}$ from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.
2. If $\Psi^{-1}(t) \lesssim \Phi^{-1}(t) t^{-\alpha/n}$, then the condition $b \in BMO(\mathbb{R}^n)$ is necessary for the boundedness of $M_{b,\alpha}$ from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.
3. If $\Phi \in \nabla_2$, $\Psi \in \Delta_2$, $\Psi^{-1}(t) \approx \Phi^{-1}(t) t^{-\alpha/n}$ and the condition

$$\sup_{r<t<\infty} \Big(1 + \ln \frac{t}{r}\Big) \Phi^{-1}\big(t^{-n}\big) t^{\alpha} \lesssim r^{\alpha} \Phi^{-1}\big(r^{-n}\big)$$

holds, then the condition $b \in BMO(\mathbb{R}^n)$ is necessary and sufficient for the boundedness of $M_{b,\alpha}$ from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

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A well known result due to Coifman, Rochberg and Weiss [CRW] (see also [S.Janson]) states that $b \in BMO(\mathbb{R}^n)$ if and only if the commutator $[b, T]$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. In 1978, Janson [S.Janson] gave some characterizations of the Lipschitz space $\dot{\Lambda}_\beta(\mathbb{R}^n)$ via commutator $[b, T]$ and proved that $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)(0 < \beta < 1)$ if and only if $[b, T]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ where $1 < p < n/\beta$ and $1/p - 1/q = \beta/n$ (see also Paluszyński [Palu]).

- R.R. Coifman, R. Rochberg, G. Weiss, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math. (2) 103 (3) (1976) 611-635.
- S. Janson, *Mean oscillation and commutators of singular integral operators*, Ark. Mat. 16 (1978) 263-270.
- M. Paluszynski, *Characterization of the Besov spaces via the commutator operator of Coifman, Rochberg and Weiss*, Indiana Univ. Math. J. (1)44 (1995), 1-17.

In this section, as an application of boundedness of fractional maximal operator on Orlicz spaces we consider the boundedness of $M_{b,\alpha}$ and $[b, M_\alpha]$ on Orlicz spaces when b belongs to the Lipschitz space, by which some new characterizations of the Lipschitz spaces are given.

Lipschitz spaces

Let $0 < \beta < 1$, we say a function b belongs to the Lipschitz space $\dot{\Lambda}_\beta(\mathbb{R}^n)$ if there exists a constant C such that for all $x, y \in \mathbb{R}^n$,

$$|b(x) - b(y)| \leq C|x - y|^\beta.$$

The smallest such constant C is called the $\dot{\Lambda}_\beta(\mathbb{R}^n)$ norm of b and is denoted by $\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}$.

Theorem B

Let $0 < \beta < 1$, $0 < \alpha < n$, $0 < \alpha + \beta < n$, $b \in L^1_{\rm loc}(\mathbb{R}^n)$, $\Phi, \Psi \in \mathcal{Y}$.
1. If $\Phi \in \nabla_2$ and the conditions

$$t^{-\frac{\alpha+\beta}{n}} \Phi^{-1}\big(t\big) \le C\Psi^{-1}\big(t\big), \tag{13}$$

and

$$\int_t^\infty \Phi^{-1}\big(r^{-n}\big) r^{\alpha+\beta} \frac{dr}{r} \le C t^{\alpha+\beta} \Phi^{-1}\big(t^{-n}\big), \tag{14}$$

hold for all $t > 0$, where $C > 0$ does not depend on t, then the condition $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ is sufficient for the boundedness of $[b, I_\alpha]$ from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.
2. If the condition

$$\Psi^{-1}(t) \le C\Phi^{-1}(t) t^{-\frac{\alpha+\beta}{n}}, \tag{15}$$

holds for all $t > 0$, where $C > 0$ does not depend on t, holds, then the condition $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ is necessary for the boundedness of $|b, I_\alpha|$ from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.
3. If $\Phi \in \nabla_2$, condition (14) holds and $\Psi^{-1}(t) \approx \Phi^{-1}(t) t^{-\frac{\alpha+\beta}{n}}$, then the condition $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ is necessary and sufficient for the boundedness of $|b, I_\alpha|$ from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

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Theorem D

Let $0 < \beta < 1$, $0 < \alpha < n$, $0 < \alpha + \beta < n$, $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $\Phi, \Psi \in \mathcal{Y}$.

1. If the conditions (13) and (14) are satisfied, then the condition $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ is sufficient for the boundedness of $[b, I_\alpha]$ from $L^\Phi(\mathbb{R}^n)$ to $WL^\Psi(\mathbb{R}^n)$.
2. If the condition (15) holds and $\frac{t^{1+\varepsilon}}{\Psi(t)}$ is almost decreasing for some $\varepsilon > 0$, then the condition $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ is necessary for the boundedness of $|b, I_\alpha|$ from $L^\Phi(\mathbb{R}^n)$ to $WL^\Psi(\mathbb{R}^n)$.
3. If $\Psi^{-1}(t) \approx \Phi^{-1}(t) t^{-\frac{\alpha+\beta}{n}}$, condition (14) holds and $\frac{t^{1+\varepsilon}}{\Psi(t)}$ is almost decreasing for some $\varepsilon > 0$, then the condition $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ is necessary and sufficient for the boundedness of $|b, I_\alpha|$ from $L^\Phi(\mathbb{R}^n)$ to $WL^\Psi(\mathbb{R}^n)$.

V.S. Guliyev, F. Deringoz, S.G. Hasanov, *Riesz potential and its commutators on Orlicz spaces*, J. Inequal. Appl. 2017, 2017:75.

To state our results, we recall the definition of the maximal operator with respect to a ball. For a fixed ball B_0, the fractional maximal function with respect to B_0 of a function f is given by

$$M_{\alpha,B_0}(f)(x) = \sup_{B_0 \supseteq B \ni x} \frac{1}{|B_0|^{1-\frac{\alpha}{n}}} \int_B |f(y)|dy, \qquad 0 \leq \alpha < n,$$

where the supremum is taken over all the balls B with $B \subseteq B_0$ and $x \in B$.

Theorem E

Let $0 < \beta < 1$, $0 \le \alpha < n$, $0 < \alpha + \beta < n$ and b be a locally integrable non-negative function. Suppose that Φ, Ψ be Young functions, $\Phi \in \mathcal{Y} \cap \nabla_2$ and $\Psi^{-1}(t) \approx \Phi^{-1}(t) t^{-\frac{\alpha+\beta}{n}}$. Then the following statements are equivalent:

1. $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$.
2. $[b, M_\alpha]$ is bounded from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.
3. There exists a constant $C > 0$ such that

$$\sup_B |B|^{-\beta/n} \Psi^{-1}\left(|B|^{-1}\right) \|b(\cdot) - |B|^{-\alpha/n} M_{\alpha,B}(b)(\cdot)\|_{L^\Psi(B)} \le C. \qquad (16)$$

V.S. Guliyev, F. Deringoz, S.G. Hasanov, *Fractional maximal function and its commutators on Orlicz spaces*, submitted.

If we take $\alpha = 0$ at Theorem E, we have the following result.

Corollary

Let $0 < \beta < 1$ and b be a locally integrable non-negative function. Suppose that Φ, Ψ be Young functions, $\Phi \in \mathcal{Y} \cap \nabla_2$ and $\Psi^{-1}(t) \approx \Phi^{-1}(t) t^{-\frac{\beta}{n}}$. Then the following statements are equivalent:

1. $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$.
2. $[b, M]$ is bounded from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.
3. There exists a constant $C > 0$ such that

$$\sup_B |B|^{-\beta/n} \Psi^{-1}\left(|B|^{-1}\right) \|b(\cdot) - M_B(b)(\cdot)\|_{L^\Psi(B)} \leq C.$$

If we take $\Phi(t) = t^p$ and $\Psi(t) = t^q$ with $1 \le p < \infty$ and $1 \le q \le \infty$ at Theorem E, we have the following result.

Corollary E

Let $0 < \beta < 1$, $0 \le \alpha < n$, $0 < \alpha + \beta < n$, $b \in L^1_{\rm loc}(\mathbb{R}^n)$, b be a locally integrable non-negative function, $1 < p < q \le \infty$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha+\beta}{n}$. Then the following statements are equivalent:

1. $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$.
2. $[b, M_\alpha]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.
3. There exists a constant $C > 0$ such that

$$\sup_B \frac{1}{|B|^{\beta/n}} \left(\frac{1}{|B|} \int_B |b(x) - |B|^{-\alpha/n} M_{\alpha,B}(b)(x)|^q dx \right)^{1/q} \le C.$$

For $\alpha = 0$, Corollary E was proved in

P. Zhang, *Characterization of Lipschitz spaces via commutators of the Hardy-Littlewood maximal function.* **C. R. Acad. Sci. Paris, Ser. I** 355 (2017), 336-344.

From the proof of Theorem E one can see that the assumption $b \geq 0$ is not used in (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1). This means (2) and (3) are sufficient conditions for $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$. But we don't know if (2) and (3) are necessary for $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$. Indeed, we have obtained the following result.

Corollary

Let $0 < \beta < 1$, $0 \leq \alpha < n$, $0 < \alpha + \beta < n$ and b be a locally integrable function. Suppose that Φ, Ψ be Young functions, $\Phi \in \mathcal{Y} \cap \nabla_2$ and $\Psi^{-1}(t) \approx \Phi^{-1}(t) t^{-\frac{\alpha+\beta}{n}}$. If one of the following statements is true, then $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$:

1. $[b, M_\alpha]$ is bounded from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.
2. There exists a constant $C > 0$ such that

$$\sup_B |B|^{-\beta/n} \Psi^{-1}\big(|B|^{-1}\big) \|b(\cdot) - |B|^{-\alpha/n} M_{\alpha,B}(b)(\cdot)\|_{L^\Psi(B)} \leq C.$$

Theorem F

Let $b \geq 0$ be a locally integrable function, $0 < \beta < 1$, $0 \leq \alpha < n$, $0 < \alpha + \beta < n$ and $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$. Suppose that Φ, Ψ be Young functions, $\Phi \in \mathcal{Y}$ and condition (13) holds. Then $[b, M_\alpha]$ is bounded from $L^\Phi(\mathbb{R}^n)$ to $WL^\Psi(\mathbb{R}^n)$.

If we take $\Phi(t) = t^p$ and $\Psi(t) = t^q$ with $1 \leq p < \infty$ and $1 \leq q \leq \infty$ at Theorem F, we have the following result.

Corollary F

Let $b \geq 0$ be a locally integrable function, $0 < \beta < 1$, $0 \leq \alpha < n$, $0 < \alpha + \beta < n$ and $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$. Suppose that Φ, Ψ be Young functions, $\Phi \in \mathcal{Y}$ and condition (13) holds. Then $[b, M_\alpha]$ is bounded from $L^\Phi(\mathbb{R}^n)$ to $WL^\Psi(\mathbb{R}^n)$.

For $\alpha = 0$, Corollary F was proved in

P. Zhang, *Characterization of Lipschitz spaces via commutators of the Hardy-Littlewood maximal function.* **C. R. Acad. Sci. Paris, Ser. I** 355 (2017), 336-344.

We find it convenient to define generalized Orlicz-Morrey spaces in the form as follows.

Definition

Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and Φ any Young function. We denote by $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ the generalized Orlicz-Morrey space, the space of all functions $f \in L^{\Phi}_{\mathrm{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{\mathcal{M}^{\Phi,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}\Phi^{-1}(|B(x,r)|^{-1})\|f\|_{L^{\Phi}(B(x,r))} < \infty.$$

Lemma

Let Φ be a Young function and φ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$.

(i) *If*

$$\sup_{t<r<\infty} \frac{\Phi^{-1}(|B(x,r)|^{-1})}{\varphi(x,r)} = \infty, \quad \textit{for some } t>0 \ \textit{ and for all } x \in \mathbb{R}^n, \tag{17}$$

then $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n) = \Theta$.

(ii) *If*

$$\sup_{0<r<\tau} \varphi(x,r)^{-1} = \infty \quad \textit{for some } \tau>0 \ \textit{ and for all } x \in \mathbb{R}^n, \tag{18}$$

then $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n) = \Theta$.

Lemma

Let Φ *be a Young function and* φ *be a positive measurable function on* $\mathbb{R}^n \times (0, \infty)$.

(i) *If*

$$\sup_{t<r<\infty} \frac{\Phi^{-1}(|B(x,r)|^{-1})}{\varphi(x,r)} = \infty, \quad \textit{for some } t > 0 \ \textit{ and for all } x \in \mathbb{R}^n, \tag{17}$$

then $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n) = \Theta$.

(ii) *If*

$$\sup_{0<r<\tau} \varphi(x,r)^{-1} = \infty \quad \textit{for some } \tau > 0 \ \textit{ and for all } x \in \mathbb{R}^n, \tag{18}$$

then $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n) = \Theta$.

Remark

Let Φ be a Young function. We denote by Ω_Φ the sets of all positive measurable functions φ on $\mathbb{R}^n \times (0,\infty)$ such that for all $t > 0$,

$$\sup_{x\in\mathbb{R}^n} \left\| \frac{\Phi^{-1}(|B(x,r)|^{-1})}{\varphi(x,r)} \right\|_{L^\infty(t,\infty)} < \infty,$$

and

$$\sup_{x\in\mathbb{R}^n} \left\| \varphi(x,r)^{-1} \right\|_{L^\infty(0,t)} < \infty,$$

respectively. In what follows, keeping in mind Lemma 2, we always assume that $\varphi \in \Omega_\Phi$.

A function $\varphi : (0, \infty) \to (0, \infty)$ is said to be almost increasing (resp. almost decreasing) if there exists a constant $C > 0$ such that

$$\varphi(r) \leq C\varphi(s) \qquad (\text{resp. } \varphi(r) \geq C\varphi(s)) \quad \text{for } r \leq s.$$

For a Young function Φ, we denote by $\mathcal{G}_\Phi$ the set of all almost decreasing functions $\varphi : (0, \infty) \to (0, \infty)$ such that $t \in (0, \infty) \mapsto \frac{1}{\Phi^{-1}(t^{-n})}\varphi(t)$ is almost increasing.

The following theorem is one of our main results.

Theorem (Adams type result for I_α)

Let $0 < \alpha < n$, $\Phi \in \mathcal{Y}$, $\varphi_1 \in \Omega_\Phi$, $\beta \in (0,1)$ and $\eta(t) \equiv \varphi(t)^\beta$ and $\Psi(t) \equiv \Phi(t^{1/\beta})$.

1. *If $\varphi(t)$ satisfies*

$$\sup_{r<t<\infty} \Phi^{-1}\left(t^{-n}\right) \operatorname*{ess\,inf}_{t<s<\infty} \frac{\varphi(s)}{\Phi^{-1}\left(s^{-n}\right)} \le C\,\varphi(r), \tag{19}$$

where C does not depend on r, then the condition

$$t^\alpha \varphi(t) + \int_t^\infty r^\alpha \varphi(r) \frac{dr}{r} \le C\varphi(t)^\beta, \tag{20}$$

for all $t > 0$, where $C > 0$ does not depend on t, is sufficient for boundedness of I_α from $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ to $W\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$.Moreover, if $\Phi \in \nabla_2$, the condition (20) is sufficient for boundedness of I_α from $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ to $W\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$ and from $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$.

Theorem

2. *If* $\varphi \in \mathcal{G}_\Phi$*, then the condition*

$$t^\alpha \varphi(t) \le C\varphi(t)^\beta, \tag{21}$$

for all $t > 0$*, where* $C > 0$ *does not depend on* t*, is necessary for boundedness of* I_α *from* $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ *to* $\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$ *and from* $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ *to* $\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$*.*

Theorem

3. *If* $\varphi \in \mathcal{G}_\Phi$ *satisfies the regularity condition*

$$\int_t^\infty r^\alpha \varphi(r) \frac{dr}{r} \le C t^\alpha \varphi(t), \tag{22}$$

for all $t > 0$*, where* $C > 0$ *does not depend on* t*, then the condition* (21) *is necessary and sufficient for boundedness of* I_α *from* $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ *to* $W\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$*. Moreover, if* $\Phi \in \nabla_2$*, the condition* (21) *is necessary and sufficient for boundedness of* I_α *from* $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ *to* $\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$*.*

Theorem

3. *If* $\varphi \in \mathcal{G}_\Phi$ *satisfies the regularity condition*

$$\int_t^\infty r^\alpha \varphi(r) \frac{dr}{r} \leq C t^\alpha \varphi(t), \tag{22}$$

for all $t > 0$*, where* $C > 0$ *does not depend on* t*, then the condition* (21) *is necessary and sufficient for boundedness of* I_α *from* $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ *to* $W\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$*. Moreover, if* $\Phi \in \nabla_2$*, the condition* (21) *is necessary and sufficient for boundedness of* I_α *from* $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ *to* $\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$*.*

[Gul 2009] V.S. Guliyev, *Boundedness of the maximal, potential and singular operators in the generalized Morrey spaces.* **Journal of Inequalities and Applications**, Volume 2009, Article ID 503948, 20 pages
If we take $\Phi(t) = t^p$, $p \in [1, \infty)$ and $\beta = \frac{p}{q}$ with $p < q < \infty$ at Theorem 3 we get the following new result for generalized Morrey spaces.

Corollary (Gul 2009)

Let $1 < p < q < \infty$.

1. *If $\varphi(t)$ satisfies*

$$\sup_{t<r<\infty} r^{-\frac{n}{p}} \operatorname*{ess\,inf}_{r<s<\infty} \varphi(s) s^{\frac{n}{p}} \leq C\varphi(t), \tag{23}$$

then the condition

$$t^{\alpha}\varphi(t) + \int_t^{\infty} r^{\alpha}\,\varphi(r)\frac{dr}{r} \leq C\varphi(t)^{\frac{p}{q}}, \tag{24}$$

for all $t > 0$, where $C > 0$ does not depend on t, is sufficient for boundedness of I_α from $\mathcal{M}^{p,\varphi}(\mathbb{R}^n)$ to $\mathcal{M}^{q,\varphi^{\frac{p}{q}}}(\mathbb{R}^n)$.

Corollary

2. *If $\varphi \in \mathcal{G}_p$, then the condition*

$$t^{\alpha}\varphi(t) \leq C\varphi(t)^{\frac{p}{q}}, \tag{25}$$

for all $t > 0$, where $C > 0$ does not depend on t, is necessary for boundedness of I_α from $\mathcal{M}^{p,\varphi}(\mathbb{R}^n)$ to $\mathcal{M}^{q,\varphi^{\frac{p}{q}}}(\mathbb{R}^n)$.

3. *If $\varphi \in \mathcal{G}_p$ satisfies the regularity condition (22), then the condition (25) is necessary and sufficient for boundedness of I_α from $\mathcal{M}^{p,\varphi}(\mathbb{R}^n)$ to $\mathcal{M}^{q,\varphi^{\frac{p}{q}}}(\mathbb{R}^n)$.*

Corollary (Adams result for I_α)

Let $0 < \alpha < n$, $1 \le p < q < \infty$ *and* $0 < \lambda < n - \alpha p$. *Then* I_α *is bounded from* $\mathcal{M}^{p,\lambda}(\mathbb{R}^n)$ *to* $W\mathcal{M}^{q,\lambda}(\mathbb{R}^n)$ *if and only if* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$. *Moreover, if* $p > 1$, *then* I_α *is bounded from* $\mathcal{M}^{p,\lambda}(\mathbb{R}^n)$ *to* $\mathcal{M}^{q,\lambda}(\mathbb{R}^n)$ *if and only if* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$.

Theorem (Adams type result for $[b, I_\alpha]$)

Let $0 < \alpha < n$, $b \in BMO(\mathbb{R}^n)$, $\Phi \in \Delta'$, $\varphi \in \Omega_\Phi$, $\beta \in (0,1)$, $\eta(t) \equiv \varphi(t)^\beta$ *and* $\Psi(t) \equiv \Phi(t^{1/\beta})$.

1. *If* $\Phi \in \nabla_2$ *and* $\varphi(t)$ *satisfies*

$$\sup_{r<t<\infty} \left(1 + \ln \frac{t}{r}\right) \Phi^{-1}\left(t^{-n}\right) \operatorname*{ess\,inf}_{t<s<\infty} \frac{\varphi(s)}{\Phi^{-1}\left(s^{-n}\right)} \leq C\,\varphi(r), \tag{26}$$

where C *does not depend on* r, *then the condition*

$$r^\alpha \varphi(r) + \int_r^\infty \left(1 + \ln \frac{t}{r}\right) \varphi(t) t^\alpha \frac{dt}{t} \leq C \varphi(r)^\beta \tag{27}$$

for all $r > 0$, *where* $C > 0$ *does not depend on* r, *is sufficient for the boundedness of* $[b, I_\alpha]$ *from* $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ *to* $\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$.

Theorem

2. *If* $\varphi \in \mathcal{G}_\Phi$, *then the condition* (21) *is necessary for the boundedness of* $|b, I_\alpha|$ *from* $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ *to* $\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$.

3. *Let* $\Phi \in \nabla_2$. *If* $\varphi \in \mathcal{G}_\Phi$ *satisfies the condition*

$$\int_r^\infty \Big(1 + \ln \frac{t}{r}\Big)\varphi(t)t^\alpha \frac{dt}{t} \le Cr^\alpha \varphi(r) \tag{28}$$

for all $r > 0$, *where* $C > 0$ *does not depend on* r, *then the condition* (21) *is necessary and sufficient for the boundedness of* $|b, I_\alpha|$ *from* $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ *to* $\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$.

Theorem

2. *If $\varphi \in \mathcal{G}_\Phi$, then the condition (21) is necessary for the boundedness of $|b, I_\alpha|$ from $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$.*

3. *Let $\Phi \in \nabla_2$. If $\varphi \in \mathcal{G}_\Phi$ satisfies the condition*

$$\int_r^\infty \Big(1 + \ln \frac{t}{r}\Big)\varphi(t)t^\alpha \frac{dt}{t} \le Cr^\alpha \varphi(r) \tag{28}$$

for all $r > 0$, where $C > 0$ does not depend on r, then the condition (21) is necessary and sufficient for the boundedness of $|b, I_\alpha|$ from $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$.

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