The extrapolation theorems for weighted generalized Morrey spaces

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(Weighted estimates)

The purpose of this talk is to present a method that allows us to obtain the boundedness of an operator

$$T : \mathcal{M}_{p,\varphi}(w) \to \mathcal{M}_{p,\varphi}(w)$$

from the boundedness of

$$T : L^p(w) \to L^p(w),$$

as soon as we have this information for a sufficiently large class of weights w

The Extrapolation theorem was first proved by Rubio de Francia

J.L. Rubio de Francia, Factorization and extrapolation of weights, Bull. Amer. Math. Soc. (N.S.), 7 (1982), 393–395.

J.L. Rubio de Francia, A new technique in the theory of A_p weights. In Topics in modern harmonic analysis, Vol. I, II (Turin/Milan, (1982), pages 571â579. Ist. Naz. Alta Mat. Francesco Severi, Rome, 1983.

J.L. Rubio de Francia, Factorization theory and A_p weights, Amer. J. Math., 106 (1984), 533–547.

- D. Cruz-Uribe, A. Fiorenza, J.M. Martell, and C. Pérez, *The boundedness of classical operators on variable $L^{p(\cdot)}$ spaces,* Ann. Acad. Sci. Fenn. Math., **31** (2006), 239–264.
- D. Cruz-Uribe, J. M. Martell and C. Pérez, *Extensions of Rubio de Francia's extrapolation theorem,* Collect. Math., **Vol. Extra** (2006), 195–231.
- D. Cruz-Uribe, J. M. Martell and C. Pérez, Weights, extrapolation and the theory of Rubio de Francia. Operator Theory: Advances and Applications, 215. Birkhäuser/Springer Basel AG, Basel, 2011.
- D. Cruz-Uribe, L. D. Wang, *Extrapolation and weighted norm inequalities in the variable Lebesgue spaces,* Trans. Amer. Math. Soc. 369 (2017), no. 2, 1205-1235.
- A.Gogatishvili and T.Kopalian, *Extensions of Rubio de Franciaâs extrapolation theorem in variable Lebesgue space and application,* Preprint arXiv:1407.5216.

Recently the extrapolation method of Rubio de Francia was extended for Morrey space in the papers

M. ROSENTHAL AND H.-J. SCHMEISSER, sl The boundedness of operators in Muckenhoupt weighted Morrey spaces via extrapolation techniques and duality. Rev. Mat. Complut. **29** (2016), 623â657.

J. DUOANDIKOETXEA AND M. ROSENTHAL *Extension and boundedness of operators on Morrey spaces from extrapolation techniques and embeddings* arXiv:1607.04565

The well-known Morrey spaces $\mathcal{M}_{p,\lambda}$ introduced by C. Morrey in 1938

C.B. MORREY, *On the solutions of quasi-linear elliptic partial differential equations,* Trans. Amer. Math. Soc., **43** (1938), 126-166.

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C.B. Morrey, *On the solutions of quasi-linear elliptic partial differential equations,* Trans. Amer. Math. Soc., **43** (1938), 126-166.

J. Peetre, *On the theory of $\mathcal{L}_{p,\lambda}$ spaces,* J. Funct. Anal., **4(** 1969), 71–87.

D. R. Adams, *A note on Riesz potentials,* Duke Math. J, **42** (1975), 765–778.

F. Chiarenza and M. Frasca, *Morrey spaces and Hardy-Littlewood maximal function,* Rend. Math. Appl., **7** (1987), 273-279.

G. DI FAZIO AND M. A. RAGUSA, *Interior estimates in Morrey spaces for strong solutions to nondivergence form equations with discontinuous coefficients,* J. Funct. Anal, **112** (1993), 24–256.

G. DI FAZIO, D. K. PALAGACHEV AND M. A. RAGUSA, *Global Morrey regularity of strong solutions to the Dirichlet problem for elliptic equations with discontinuous coeficients,* J. Funct. Anal, **166** (1999), 179–196.

D. S. FAN, S. Z. LU AND D. C. YANG, *Regularity in Morrey spaces of strong solutions to nondivergence elliptic equations with VMO coefficients,* Georgian Math. J, **5** (1998), 425–440.

- *V. S. Guliyev,* Function Spaces, Integral Operators and Two Weighted Inequalities on Homogeneous Groups. Some Applications, Baku, 1996.
- V. S. GULIYEV, Integral Operators on Function Spaces on the Homogeneous Groups and on Domains in $\mathbb{R}^n$. Doctoral Degree Dissertation. Mat. Inst. Steklov, Moskva, 1994. (In Russian.)

- *V. S. Guliyev,* Function Spaces, Integral Operators and Two Weighted Inequalities on Homogeneous Groups. Some Applications, Baku, 1996.
- V. S. GULIYEV, Integral Operators on Function Spaces on the Homogeneous Groups and on Domains in $\mathbb{R}^n$. Doctoral Degree Dissertation. Mat. Inst. Steklov, Moskva, 1994. (In Russian.)
- V. I. BURENKOV, *Recent progress in studying the boundedness of classical operators of real analysis in general Morrey-type spaces. I.* Eurasian Math. J. **3** (2012), 11–32.
- V. I. BURENKOV, *Recent progress in studying the boundedness of classical operators of real analysis in general Morrey-type spaces. II.* Eurasian Math. J. **4** (2013), 21–45.

- A weight is a locally integrable function on $\mathbb{R}^n$ which takes values in $(0, \infty)$ almost everywhere.
- For a weight w and a measurable set E, we denote $w(E) = \int_E w(x)dx$,
- $|E|$ - is the Lebesgue measure of E.
- χ_E - is characteristic function of E. $B(a, r)$ -is the open ball in $\mathbb{R}^n$ centred at a with the radius r, $a \in \mathbb{R}^n$ and $r > 0$.
- The weighted Lebesgue spaces with respect to the measure $w(x)dx$ are denoted by $L_{p,w}(\mathbb{R}^n)$ with $0 < p < \infty$, and

$$\|f\|_{p,w} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)dx \right)^{\frac{1}{p}}.$$

we put $p' = \frac{p}{p-1}$, if $1 \leq p < \infty$.

Definition

Let $1 \le p < \infty$. Let be φ a positive measurable function on $\mathbb{R}^n \times (0, \infty)$, non-decreasing with respect to r for every $x \in \mathbb{R}^n$ and w weighted function defined on $\mathbb{R}^n$. We define a weighted generalized Morrey space $\mathcal{M}_{p,\varphi}(w)$ by

$$\mathcal{M}_{p,\varphi}(w) := \left\{ f \in L^{\mathrm{loc}}_{p,w}(\mathbb{R}^n) : \quad \|f\|_{\mathcal{M}_{p,\varphi}(w)} < \infty \right\},$$

where

$$\|f\|_{\mathcal{M}_{p,\varphi}(w)} := \sup_{x \in \mathbb{R}^n,\ r>0} \left(\frac{1}{\varphi(x,r)} \int_{B(x,r)} |f(y)|^p w(y) dy \right)^{\frac{1}{p}}.$$

If $w \equiv 1$, and $\varphi(x, r) = \psi(r)$ for every $x \in \mathbb{R}^n$, where $\psi : (0, \infty) \to (0, \infty)$ is a non-decreasing function we obtain generalized Morey space $\mathcal{M}_{p,\psi}$

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If $\varphi(x,r) = w(B(x,r))^\kappa$, $0 < \kappa < 1$ we get weighted Morrey spaces $\mathcal{M}_{p,\kappa}(w)$, consider in

Y. Komori and S.Shirai, *Weighted Morrey spaces and a singular integral operator,* Math. Nachr., **282** (2009), 219–231.

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Y. KOMORI AND S.SHIRAI, *Weighted Morrey spaces and a singular integral operator,* Math. Nachr., **282** (2009), 219–231.

For more properties about the weighted generalized Morrey space we refer the reader to

V. S.GULIYEV, *Generalized weighted Morrey spaces and higher order commutators of sublinear operators,* Eurasian Math. J., **3** (2012), 33–61.

Muckenhoupt A_p weights,
$w \in A_p$, $1 \leq p \leq \infty$, if there exists a constant C such that for every balls $B \subset \mathbb{R}^n$ we have

$$\frac{1}{|B|}\int_B w(x)dx \left(\frac{1}{|B|}\int_B w(x)^{1-p'}dx\right)^{p-1} \leq C,$$

when $1 < p < \infty$, and for $p = 1$

$$\frac{1}{|B|}\int_B w(x)dx \leq C \operatorname*{ess\,inf}_{x\in B} w(x).$$

The class A_∞ is defined as $A_\infty = \cup_{p\geq 1} A_p$.

We will consider following weighted norm inequalities

$$\int_{\mathbb{R}^n} Tf(x)^{p_0} w^{\delta}(x)dx \le C_0 \int_{\mathbb{R}^n} f(x)^{p_0} w^{\delta}(x)dx, \tag{1.1}$$

where T is some operator and $w \in A_1$, $0 < p_0 < \infty$, $0 < \delta \le 1$. (In other words, T is defined and bounded on $L_{p_0}(w^{\delta})$.) The constant C_0 is assumed to depend only on the A_1 constant of w.

Hereafter $\mathcal{F}$ will denote a family of pairs (f, g) of non-negative, measurable functions on $\mathbb{R}^n$. We say that an inequality

$$\int_{\mathbb{R}^n} f(x)^p w(x) dx \leq C \int_{\mathbb{R}^n} g(x)^p w(x) dx$$

holds for any $(f, g) \in \mathcal{F}$ and $w \in A_q$ (for some $q,\ 1 \leq q < \infty$), we mean that it holds for any pair in $\mathcal{F}$ such that the left-hand side is finite, and the constant C depends only p and the A_q constant of w.

Theorem

Given a family $\mathcal{F}$, *suppose that for some* $p, \delta, 0 < p < \infty, 0 < \delta \leq 1$, *and for every weight* $w \in A_1$

$$\int_{\mathbb{R}^n} f(x)^p w^\delta(x) dx \leq C_0 \int_{\mathbb{R}^n} g(x)^p w^\delta(x) dx, \quad (f, g) \in \mathcal{F}. \tag{2.1}$$

Let $w \in A_1$ *such that*

$$\sum_{k=1}^{\infty} \frac{\varphi(x, 2^k r)|B(x, 2^k r)|^{\varepsilon_0}}{w(B(x, 2^k r))^\delta} \leq C_1 \frac{\varphi(x, r)|B(x, r)|^{\varepsilon_0}}{w(B(x, r))^\delta} \tag{2.2}$$

for every $x \in \mathbb{R}^n$ *and* $r > 0$, *and for some* $\varepsilon_0 > 0$. *Then for all* $(f, g) \in \mathcal{F}$

$$\|f\|_{\mathcal{M}_{p,\varphi}(w^\delta)} \leq C_2 \|g\|_{\mathcal{M}_{p,\varphi}(w^\delta)}. \tag{2.3}$$

Theorem

Given a family $\mathcal{F}$, assume that for some p q and δ, $0 < p \le q < \infty$, $0 < \delta \le 1$ and every weight $w \in A_1$,

$$\left(\int_{\mathbb{R}^n} f(x)^q w(x)^\delta dx\right)^{1/q} \le C_0 \left(\int_{\mathbb{R}^n} g(x)^p w(x)^{\frac{\delta p}{q}} dx\right)^{1/p}, \quad (f,g) \in \mathcal{F}. \qquad (2.4)$$

Let $w \in A_1$ such that

$$\sum_{k=1}^{\infty} \frac{\varphi(x,2^k r)^{\frac{p}{q}} |B(x,2^k r)|^{\varepsilon_0}}{w(B(x,2^k r))^{\frac{\delta p}{q}}} \le C_1 \frac{\varphi(x,r)^{\frac{p}{q}} |B(x,r)|^{\varepsilon_0}}{w(B(x,r))^{\frac{\delta p}{q}}} \qquad (2.5)$$

for every $x \in \mathbb{R}^n$ and $r > 0$, and for some $\varepsilon_0 > 0$. Then for all $(f,g) \in \mathcal{F}$

$$\|f\|_{\mathcal{M}_{q,\varphi}(w^\delta)} \le C_2 \|g\|_{\mathcal{M}_{p,\varphi^{p/q}}(w^{\frac{\delta p}{q}})}. \qquad (2.6)$$

Uusing rescaling argument we obtain following

Corollary

Suppose that for some p_0, $0 < p_0 < \infty$, *the family* $\mathcal{F}$ *is such that for all* $w \in A_1$

$$\int_{\mathbb{R}^n} f(x)^{p_0} w(x)dx \le C_0 \int_{\mathbb{R}^n} g(x)^{p_0} w(x)dx, \quad (f,g) \in \mathcal{F}. \tag{2.7}$$

Given φ *and* $w \in A_1$, *such that*

$$\sum_{k=1}^{\infty} \frac{\varphi(x, 2^k r)|B(x, 2^k r)|^{\varepsilon_0}}{w(B(x, 2^k r))} \le C_1 \frac{\varphi(x, r)|B(x, r)|^{\varepsilon_0}}{w(B(x, r))} \tag{2.8}$$

for every $x \in \mathbb{R}^n$ *and* $r > 0$, *and for some* $\varepsilon_0 > 0$. *Suppose* $p_0 < p < \infty$. *Then for all* $(f, g) \in \mathcal{F}$

$$\|f\|_{\mathcal{M}_{p,\varphi}(w)} \le C_2 \|g\|_{\mathcal{M}_{p,\varphi}(w)}. \tag{2.9}$$

Theorem

Assume that $0 < p \le q < \infty,\ 0 < \delta \le 1$. *Let* $w \in A_1$ *such that*

$$\sum_{k=1}^{\infty} \frac{\varphi(x,2^k r)^{\frac{p}{q}} |B(x,2^k r)|^{\varepsilon_0}}{w(B(x,2^k r))^{\frac{\delta p}{q}}} \le C_0 \frac{\varphi(x,r)^{\frac{p}{q}} |B(x,r)|^{\varepsilon_0}}{w(B(x,r))^{\frac{\delta p}{q}}}$$

for every $x \in \mathbb{R}^n$ *and* $r > 0$, *and for some* $\varepsilon_0 > 0$. *Then there exists a constant* $C_1 > 0$ *and* $s > 1$ *such that for all* $f \in \mathcal{M}_{p,\varphi^{\frac{p}{q}}}(w^{\frac{\delta p}{q}})$, *and all ball* $B(x_0, r)$ *holds*

$$\left(\int_{\mathbb{R}^n} |f(x)|^p (M(\chi_{B(x_0,r)} w^s)(x))^{\frac{\delta p}{sq}} dx \right)^{\frac{1}{p}} \le C_1 \varphi(x_0,r)^{\frac{1}{q}} \|f\|_{\mathcal{M}_{p,\varphi^{\frac{p}{q}}}(w^{\frac{\delta p}{q}})}. \quad (2.10)$$

If we take $p = q$ we get following result

Theorem

Assume that $0 < p < \infty,\ 0 < \delta \leq 1$. *Let* $w \in A_1$ *such that*

$$\sum_{k=1}^{\infty} \frac{\varphi(x, 2^k r)|B(x, 2^k r)|^{\varepsilon_0}}{w(B(x, 2^k r))^{\delta}} \leq C_1 \frac{\varphi(x, r)|B(x, r)|^{\varepsilon_0}}{w(B(x, r))^{\delta}}$$

for every $x \in \mathbb{R}^n$ *and* $r > 0$, *and for some* $\varepsilon_0 > 0$. *Then there exists a constant* $C_1 > 0$ *and* $s > 1$ *such that for all* $f \in \mathcal{M}_{p,\varphi}(w^{\delta})$, *and all ball* $B(x_0, r)$ *holds*

$$\left(\int_{\mathbb{R}^n} |f(x)|^p (M(\chi_{B(x_0,r)} w^s)(x))^{\frac{\delta}{s}} dx \right)^{\frac{1}{p}} \leq C_1 \varphi(x_0, r)^{\frac{1}{p}} \|f\|_{\mathcal{M}_{p,\varphi}(w^{\delta})}.$$

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$$\sum_{k=1}^{\infty} \frac{\varphi(x, 2^k r)|B(x, 2^k r)|^{\varepsilon_0}}{w(B(x, 2^k r))^{\delta}} \leq C_1 \frac{\varphi(x, r)|B(x, r)|^{\varepsilon_0}}{w(B(x, r))^{\delta}}$$

for every $x \in \mathbb{R}^n$ *and* $r > 0$, *and for some* $\varepsilon_0 > 0$. *Then there exists a constant* $C_1 > 0$ *and* $s > 1$ *such that for all* $f \in \mathcal{M}_{p,\varphi}(w^{\delta})$, *and all ball* $B(x_0, r)$ *holds*

$$\left(\int_{\mathbb{R}^n} |f(x)|^p (M(\chi_{B(x_0,r)} w^s)(x))^{\frac{\delta}{s}} dx\right)^{\frac{1}{p}} \leq C_1 \varphi(x_0, r)^{\frac{1}{p}} \|f\|_{\mathcal{M}_{p,\varphi}(w^{\delta})}.$$

Therefore, we have the inequalities of the form (1.1) with weights $v(x) = (M(\chi_{B(x_0,r)} w^s)(x))^{\frac{1}{s}}$ for some $s > 1$. As the family $\mathcal{F}$ in the hypothesis and conclusion of Theorem is the same, so the goal is to find a large, reasonable family $\mathcal{F}$ such that (1.1) holds with a constant depending only on p_0 and the A_1 constant of w.

Theorem

Given a operator T, suppose that for some p, q and δ $0 < p \le q < \infty$, $0 < \delta \le 1$ and every $w \in A_1$, there exists constant C_0 depending on A_1 constant of w such that

$$\left(\int_{\mathbb{R}^n} |Tf(x)|^q w(x)^\delta dx\right)^{\frac{1}{q}} \le C_0 \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)^{\frac{\delta p}{q}} dx\right)^{\frac{1}{p}}, \quad f \in L_{p,w^{\frac{\delta p}{q}}}.$$

Given φ and $w \in A_1$, such that

$$\sum_{k=1}^{\infty} \frac{\varphi(x,2^k r)^{\frac{p}{q}} |B(x,2^k r)|^{\varepsilon_0}}{w(B(x,2^k r))^{\frac{\delta p}{q}}} \le C_1 \frac{\varphi(x,r)^{\frac{p}{q}} |B(x,r)|^{\varepsilon_0}}{w(B(x,r))^{\frac{\delta p}{q}}} \tag{2.11}$$

for every $x \in \mathbb{R}^n$ and $r > 0$, and for some $\varepsilon_0 > 0$. Then there exists constant C_1 such that for all $f \in \mathcal{M}_{p,\varphi^{\frac{p}{q}}}(w^{\frac{\delta p}{q}})$

$$\|Tf\|_{\mathcal{M}_{q,\varphi}(w^\delta)} \le C_2 \|f\|_{\mathcal{M}_{p,\varphi^{\frac{p}{q}}}(w^{\frac{\delta p}{q}})}. \tag{2.12}$$

Theorem

Given a operators T, S suppose that for some p, q, δ, $0 < p \leq q < \infty$, $0 < \delta \leq 1$ and every $w \in A_1$, there exists constant C_0 depending on A_1 constant of w such that

$$\left(\int_{\mathbb{R}^n} |Tf(x)|^q w(x)^\delta dx \right)^{1/q} \leq C_0 \left(\int_{\mathbb{R}^n} |Sf(x)|^p w(x)^{\frac{\delta p}{q}} dx \right)^{1/p}, \quad Sf \in L_{p, w^{\frac{\delta p}{q}}}.$$

Given φ and $w \in A_1$, such that

$$\sum_{k=1}^{\infty} \frac{\varphi(x, 2^k r)^{\frac{p}{q}} |B(x, 2^k r)|^{\varepsilon_0}}{w(B(x, 2^k r))^{\frac{\delta p}{q}}} \leq C_1 \frac{\varphi(x, r)^{\frac{p}{q}} |B(x, r)|^{\varepsilon_0}}{w(B(x, r))^{\frac{\delta p}{q}}} \tag{2.13}$$

for every $x \in \mathbb{R}^n$ and $r > 0$, and for some $\varepsilon_0 > 0$. Then there exists constant C_1 such that for all $Sf \in \mathcal{M}_{p, \varphi^{\frac{p}{q}}}(w^{\frac{\delta p}{q}})$

$$\|Tf\|_{\mathcal{M}_{q,\varphi}(w^\delta)} \leq C_2 \|Sf\|_{\mathcal{M}_{p, \varphi^{\frac{p}{q}}}(w^{\frac{\delta p}{q}})}. \tag{2.14}$$

Theorem

Given a operator T, S *suppose that for some* p, δ, $0 < p < \infty$, $0 < \delta \leq 1$ *and every* $w \in A_1$, *there exists constant* C_0 *depending on* A_1 *constant of* w *such that*

$$\int_{\mathbb{R}^n} |Tf(x)|^p w(x)^\delta dx \leq C_0 \int_{\mathbb{R}^n} |Sf(x)|^p w(x)^\delta dx, \quad Sf \in L_{p,w^\delta}.$$

Given φ *and* $w \in A_1$, *such that*

$$\sum_{k=1}^{\infty} \frac{\varphi(x, 2^k r)|B(x, 2^k r)|^{\varepsilon_0}}{w(B(x, 2^k r))^\delta} \leq C_1 \frac{\varphi(x, r)|B(x, r)|^{\varepsilon_0}}{w(B(x, r))^\delta} \tag{2.15}$$

for every $x \in \mathbb{R}^n$ *and* $r > 0$, *and for some* $\varepsilon_0 > 0$. *Then there exists constant* C_1 *such that for all* $Sf \in \mathcal{M}_{p,\varphi}(w^\delta)$

$$\|Tf\|_{\mathcal{M}_{p,\varphi}(w^\delta)} \leq C_2 \|Sf\|_{\mathcal{M}_{p,\varphi}(w^\delta)}.$$

Define the Spherical Maximal operator $\mathcal{M}$, by

$$\mathcal{M}f(x) := \sup_{t>0} |\mu_t * f(x)|,$$

where μ_t denotes the normalized surface measure on the sphere of center 0 and radius t in $\mathbb{R}^n$.

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Theorem

If $n > 2$, $\frac{n}{n-1} < \sigma < \infty$ and w belongs to the A_1 Muckenhoupt class, then

$$\|\mathcal{M}f\|_{\sigma,w^\delta} \lesssim C\|f\|_{\sigma,w^\delta}, \tag{4.1}$$

where $\delta = \frac{n-2}{n-1}$.

M. Cowling, J. García-Cuerva and H. Gunawan, *Weighted estimates for fractional maximal functions related to spherical maens,* Bull. Austral. Math. Soc., **66** (2002), 75–90.

Theorem

Let $n > 2$, $\frac{n}{n-1} < p < \infty$ *and* $\delta = \frac{n-2}{n-1}$. *Let be* $w \in A_1$ *such that*

$$\sum_{k=1}^{\infty} \frac{\varphi(x, 2^k r)|B(x, 2^k r)|^{\varepsilon}}{w(B(x, 2^k r))^{\delta}} \leq C_1 \frac{\varphi(x, r)|B(x, r)|^{\varepsilon}}{w(B(x, r))^{\delta}} \tag{4.2}$$

for every $x \in \mathbb{R}^n$ *and* $r > 0$, *and for some* $\varepsilon > 0$. *Then*

$$\|\mathcal{M}f\|_{\mathcal{M}_{p,\varphi}(w^{\delta})} \leq C\|f\|_{\mathcal{M}_{p,\varphi}(w^{\delta})}. \tag{4.3}$$

Corollary

Let $n > 2$, $\frac{n}{n-1} < p < \infty$, $0 < \kappa < \frac{n-2}{n-1}$, $\delta = \frac{n-2}{n-1}$, *and* $w \in A_1$. *Then*

$$\|\mathcal{M}f\|_{\mathcal{M}_{p,\kappa}(w^\delta)} \le C\|f\|_{\mathcal{M}_{p,\kappa}(w^\delta)}. \tag{4.4}$$

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Let $n > 2$, $\frac{n}{n-1} < p < \infty$, $0 < \kappa < \frac{n-2}{n-1}$, $\delta = \frac{n-2}{n-1}$, *and* $w \in A_1$. *Then*

$$\|\mathcal{M}f\|_{\mathcal{M}_{p,\kappa}(w^\delta)} \le C\|f\|_{\mathcal{M}_{p,\kappa}(w^\delta)}. \tag{4.4}$$

If $w \equiv 1$ we get

Corollary

Let $n > 2$, $\frac{n}{n-1} < p < \infty$, $0 < \lambda < n\frac{n-2}{n-1}$. *Then*

$$\|\mathcal{M}f\|_{\mathcal{M}_{p,\lambda}} \le C\|f\|_{\mathcal{M}_{p,\lambda}}. \tag{4.5}$$

Denote by μ_t the normalized surface measure on the sphere in $\mathbb{R}^n$ with center 0 and radius t. The maximal operator related to spherical means is given by

$$\mathcal{M}^{\alpha} = \sup_{t>0} |t^{\alpha} \mu_t * f|.$$

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The weighted $L^p \to L^q$ estimate for the maximal operators $\mathcal{M}^\alpha$, was invesigated in the paper

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Theorem

Suppose that $n > 2$, $\frac{n}{n-1} < p < q < n$, *that* $\alpha = n/p - n/q$, *and that* $\max\{0, 1 - q/p'\} < \gamma \leq 1 - q/n$. *Suppose also that* w *is in* A_s, *where*

$$s = \frac{q + 2p'\gamma - p'}{p'\gamma}.$$

Then there exists a constant C *such that*

$$\|\mathcal{M}^\alpha f\|_{q,w^\gamma} \leq C\|f\|_{p,w^{\frac{\gamma p}{q}}}.$$

Theorem

Let $n > 2$, $\frac{n}{n-1} < p < q < n$, $\alpha = n/p - n/q$, *and* $\max\{0, 1 - q/p'\} < \gamma \leq 1 - q/n$. *Suppose also that* w *is in* A_1 *for which*

$$\sum_{k=1}^{\infty} \frac{\varphi(x, 2^k r)^{\frac{p}{q}} |B(x, 2^k r)|^{\varepsilon}}{w(B(x, 2^k r))^{\frac{\gamma p}{q}}} \leq C_1 \frac{\varphi(x, r)^{\frac{p}{q}} |B(x, r)|^{\varepsilon}}{w(B(x, r))^{\frac{\gamma p}{q}}} \tag{7.1}$$

for every $x \in \mathbb{R}^n$ *and* $r > 0$, *and for some* $\varepsilon > 0$. *Then*

$$\|\mathcal{M}^{\alpha} f\|_{\mathcal{M}_{q,\varphi}(w^{\gamma})} \leq C \|f\|_{\mathcal{M}_{p,\varphi^{p/q}}(w^{\frac{\gamma p}{q}})}. \tag{7.2}$$

Corollary

Let $n > 2$, $\frac{n}{n-1} < p < q < n$, $\alpha = n/p - n/q$, $\max\{0, 1 - q/p'\} < \gamma \leq 1 - q/n$, $0 < \kappa < \gamma$. *Suppose that* $w \in A_1$. *Then*

$$\|\mathcal{M}^{\alpha} f\|_{\mathcal{M}_{q,\kappa}(w^{\gamma})} \leq C\|f\|_{\mathcal{M}_{p,\kappa}(w^{p\gamma/q})}.$$

Corollary

Let $n > 2$, $\frac{n}{n-1} < p < q < n$, $\alpha = n/p - n/q$, $\max\{0, 1 - q/p'\} < \gamma \leq 1 - q/n$, $0 < \kappa < \gamma$. *Suppose that* $w \in A_1$. *Then*

$$\|\mathcal{M}^{\alpha} f\|_{\mathcal{M}_{q,\kappa}(w^{\gamma})} \leq C \|f\|_{\mathcal{M}_{p,\kappa}(w^{p\gamma/q})}.$$

Corollary

Let $n > 2$, $\frac{n}{n-1} < p < q < n$, $\alpha = n/p - n/q$, $0 < \lambda < n - q$. *Then*

$$\|\mathcal{M}^{\alpha} f\|_{\mathcal{M}_{q,\lambda}} \leq C \|f\|_{\mathcal{M}_{p,\frac{\lambda p}{q}}}.$$

Theorem (D.K. Watson)

Let $n \geq 2$, $1 < r < \infty$ and let $Tf(x) = p.v.K * f(x)$ be singular integral operator with "rough" kernel

$$K(x) = h(|x|)\frac{\Omega(x)}{|x|^n},$$

where Ω is homogeneous of degree 0 on $\mathbb{R}^n$, $\Omega \in L^r(S^{n-1})$, where S^{n-1} denote the unit sphere in $\mathbb{R}^n$. Ω has average 0 on S^{n-1}, and h is a measurable function on $(0, \infty)$ satisfying

$$\int_R^{2R} |h(t)|^r dt \leq CR \quad \text{for all } R > 0.$$

Then T is bounded on $L_{p,w}(\mathbb{R}^n)$,

$$\|Tf\|_{p,w} \leq C\|f\|_{p,w},$$

in each of the following situations:
(A) if $r' \leq p < \infty$, and $w \in A_{\frac{p}{r'}}$, or
(B) if $1 < p \leq r, p \neq \infty$ and $w^{\frac{-1}{p-1}} \in A_{\frac{p'}{r'}}$, or
(C) if $1 < p < \infty$ and $w^{r'} \in A_p$.

D.K. Watson, *Weighted estimates for singular integrals via Fourier transform estimates,* Duke Math. J., **60** (1990), 389–399.

Theorem

Let $1 < r < \infty$ *and* $r' \leq p < \infty$. *Suppose that* w *is in* A_1 *for which*

$$\sum_{k=1}^{\infty} \frac{\varphi(x, 2^k t)|B(x, 2^k t)|^{\varepsilon}}{w(B(x, 2^k t))} \leq C_1 \frac{\varphi(x, t)|B(x, t)|^{\varepsilon}}{w(B(x, t))} \tag{8.1}$$

for every $x \in \mathbb{R}^n$ *and* $t > 0$, *and for some* $\varepsilon > 0$. *Then*

$$\|Tf\|_{\mathcal{M}_{p,\varphi}(w)} \leq C\|f\|_{\mathcal{M}_{p,\varphi}(w)}. \tag{8.2}$$

Theorem

Let $1 < r < \infty$ *and* $r' \leq p < \infty$. *Suppose that* w *is in* A_1 *for which*

$$\sum_{k=1}^{\infty} \frac{\varphi(x,2^k t)|B(x,2^k t)|^{\varepsilon}}{w(B(x,2^k t))} \leq C_1 \frac{\varphi(x,t)|B(x,t)|^{\varepsilon}}{w(B(x,t))} \tag{8.1}$$

for every $x \in \mathbb{R}^n$ *and* $t > 0$, *and for some* $\varepsilon > 0$. *Then*

$$\|Tf\|_{\mathcal{M}_{p,\varphi}(w)} \leq C\|f\|_{\mathcal{M}_{p,\varphi}(w)}. \tag{8.2}$$

Corollary

Let $1 < r < \infty$ *and* $r' \leq p < \infty$. *Suppose also that* w *is in* A_1 *and* $0 < \kappa < 1$. *Then*

$$\|Tf\|_{\mathcal{M}_{p,\kappa}(w)} \leq C\|f\|_{\mathcal{M}_{p,\kappa}(w)}.$$

Theorem

Let $1 < r < \infty$ *and* $p < r$. *Suppose also that* w *is in* A_1 *for which*

$$\sum_{k=1}^{\infty} \frac{\varphi(x,2^k t)|B(x,2^k t)|^{\varepsilon}}{w(B(x,2^k t))^{\frac{r-p}{r}}} \leq C_1 \frac{\varphi(x,t)|B(x,t)|^{\varepsilon}}{w(B(x,t))^{\frac{r-p}{r}}} \tag{8.3}$$

for every $x \in \mathbb{R}^n$ *and* $t > 0$, *and for some* $\varepsilon > 0$. *Then*

$$\|Tf\|_{\mathcal{M}_{p,\varphi}(w^{\frac{r-p}{r}})} \leq C\|f\|_{\mathcal{M}_{p,\varphi}(w^{\frac{r-p}{r}})}. \tag{8.4}$$

Theorem

Let $1 < r < \infty$ *and* $p < r$*. Suppose also that* w *is in* A_1 *for which*

$$\sum_{k=1}^{\infty} \frac{\varphi(x, 2^k t)|B(x, 2^k t)|^{\varepsilon}}{w(B(x, 2^k t))^{\frac{r-p}{r}}} \le C_1 \frac{\varphi(x, t)|B(x, t)|^{\varepsilon}}{w(B(x, t))^{\frac{r-p}{r}}} \tag{8.3}$$

for every $x \in \mathbb{R}^n$ *and* $t > 0$*, and for some* $\varepsilon > 0$*. Then*

$$\|Tf\|_{\mathcal{M}_{p,\varphi}(w^{\frac{r-p}{r}})} \le C\|f\|_{\mathcal{M}_{p,\varphi}(w^{\frac{r-p}{r}})}. \tag{8.4}$$

Corollary

Let $1 < r < \infty$ *and* $p < r$*. Suppose that* w *is in* A_1 *and* $0 < \kappa < \frac{r-p}{r}$*. Then*

$$\|Tf\|_{\mathcal{M}_{p,\kappa}(w^{\frac{r-p}{r}})} \le C\|f\|_{\mathcal{M}_{p,\kappa}(w^{\frac{r-p}{r}})}.$$

Theorem

Let $1 < r, p < \infty$. *Suppose that* w *is in* A_1 *for which*

$$\sum_{k=1}^{\infty} \frac{\varphi(x, 2^k t)|B(x, 2^k t)|^{\varepsilon}}{w(B(x, 2^k t))^{\frac{r-1}{r}}} \leq C_1 \frac{\varphi(x, t)|B(x, t)|^{\varepsilon}}{w(B(x, t))^{\frac{r-1}{r}}} \tag{8.5}$$

for every $x \in \mathbb{R}^n$ *and* $t > 0$, *and for some* $\varepsilon > 0$. *Then*

$$\|Tf\|_{\mathcal{M}_{p,\varphi}(w^{\frac{r-1}{r}})} \leq C\|f\|_{\mathcal{M}_{p,\varphi}(w^{\frac{r-1}{r}})}. \tag{8.6}$$

Theorem

Let $1 < r, p < \infty$. *Suppose that* w *is in* A_1 *for which*

$$\sum_{k=1}^{\infty} \frac{\varphi(x, 2^k t)|B(x, 2^k t)|^{\varepsilon}}{w(B(x, 2^k t))^{\frac{r-1}{r}}} \leq C_1 \frac{\varphi(x, t)|B(x, t)|^{\varepsilon}}{w(B(x, t))^{\frac{r-1}{r}}} \tag{8.5}$$

for every $x \in \mathbb{R}^n$ *and* $t > 0$, *and for some* $\varepsilon > 0$. *Then*

$$\|Tf\|_{\mathcal{M}_{p,\varphi}(w^{\frac{r-1}{r}})} \leq C\|f\|_{\mathcal{M}_{p,\varphi}(w^{\frac{r-1}{r}})}. \tag{8.6}$$

Corollary

Let $1 < r, p < \infty$. *Suppose that* w *is in* A_1 *and* $0 < \kappa < \frac{r-1}{r}$. *Then*

$$\|Tf\|_{\mathcal{M}_{p,\kappa}(w^{\frac{r-1}{r}})} \leq C\|f\|_{\mathcal{M}_{p,\kappa}(w^{\frac{r-1}{r}})}.$$

Let $\theta(\xi)$ be a smooth radial cut-off function $\theta(\xi) = 1$ if $|\xi| \geq 1$ and $\theta(\xi) = 0$ if $|\xi| \leq 1/2$. We will consider the multipliers

$$\widehat{T_{b,a}f}(\xi) = \theta(\xi)\frac{e^{i|\xi|^b}}{|\xi|^a}\hat{f}(\xi),$$

where $0 < b < 1$ and $0 < a < nb/2$. C. Fefferman proved that if $0 < a < nb/2$, and p is such that $|1/p - 1/2| \leq a/nb$, then

$$\|T_{b,a}\|_p \leq c_p\|f\|_p.$$

C. Fefferman, *Inequalities for strongly singular convolution operators,* Acta Math., **124** (1970), 9–36.

The weighted extension of Fefferman's theorem was obtained by Chanillo,

Theorem

Let $1 < p < \infty$, $\alpha = nb|1/p - 1/2|$, and $w \in A_p$. Then for $\alpha \leq a \leq nb/2$, and for γ, such that $\gamma = (a - \alpha)/(nb/2 - \alpha)$ we have

$$\|T_{b,a}f\|_{p,w^\gamma} \leq C_p\|f\|_{p,w^\gamma}. \tag{9.1}$$

S. Chanillo, *Weighted norm inequalities for strongly singular convolution operators,* Trans. Amer. Math. Soc., **281** (1984), 77–107.

Theorem

Let for some $1 < p < \infty$, $\alpha = nb|1/p - 1/2|$, $\alpha < a \leq nb/2$,. *Let* $\gamma = (a - \alpha)/(nb/2 - \alpha)$ *and*

$$\sum_{k=1}^{\infty} \frac{\varphi(x, 2^k r)|B(x, 2^k r)|^{\varepsilon}}{w(B(x, 2^k r))^{\gamma}} \leq C_1 \frac{\varphi(x, r)|B(x, r)|^{\varepsilon}}{w(B(x, r))^{\gamma}}$$

for every $x \in \mathbb{R}^n$ *and* $r > 0$, *and for some* $\varepsilon > 0$. *Then operator* $T_{b,a}$ *is bounded on* $\mathcal{M}_{p,\varphi}(w^{\gamma})$.

Corollary

Let for some $1 < p < \infty$, $\alpha = nb|1/p - 1/2|$, $\alpha < a \leq nb/2$,. *Let* $\gamma = (a - \alpha)/(nb/2 - \alpha)$ *and* $0 < \kappa < \gamma$. *Then operator* $T_{b,a}$ *is bounded on* $\mathcal{M}_{p,\kappa}(w^{\gamma})$.

Corollary

Let for some $1 < p < \infty$, $\alpha = nb|1/p - 1/2|$, $\alpha < a \leq nb/2$, *and* $0 < \lambda < n(a - \alpha)/(nb/2 - \alpha)$. *Then operator* $T_{b,a}$ *is bounded on* $\mathcal{M}_{p,\lambda}$.

The Bochner-Riesz operator in $\mathbb{R}^n$, $(n \geq 2)$ are defined for $\beta > 0$, as

$$\widehat{T^r_\beta}(\xi) = \left(1 - \frac{|\xi|^2}{r^2}\right)^\beta_+ \widehat{f}(\xi)$$

with $t_+ = \max(t, 0)$, and the maximal Bochner-Riesz operator is defined by

$$T^*_\beta f(x) = \sup_{r>0} |T^r_\beta f(x)|.$$

Theorem

If $0 < \beta < \frac{n-1}{2}$, then T^*_β is bounded on $L^2(w^{\frac{2\beta}{n-1}})$ for $w \in A_2$.

M. Carro, J. Duoandikoetxea and M. Lorente, *Weighted estimates in a limited range with applications to the Bochner-Riesz operators,* Indiana Univ. Math. J., **61** (2012), 1485–1511.

The following theorem in case $p = 2$ was proved by Duoandikoetxea, *et. al*. in

J. DUOANDIKOETXEA, A. MOYUA, O. ORUETXEBARRIA, AND E. SIEJO, *Radial A_p weights with applications to the disc multiplier and the Bochner-Riesz operators,* Indiana Univ. Math. J., **57** (2008), 1239–1258.

For the case $p \neq 2$

Theorem

Given δ, $0 < \delta < 1$, suppose that for all $w \in A_2$

$$\int_{\mathbb{R}^n} f(x)^2 w^\delta(x)dx \leq C \int_{\mathbb{R}^n} g(x)^2 w^\delta(x)dx, \quad (f,g) \in \mathcal{F}.$$

Then for all p, $\frac{2}{1+\delta} < p < \frac{2}{1-\delta}$, and every $w^{\frac{2}{2-p(1-\delta)}} \in A_{\frac{2p\delta}{2-p(1-\delta)}}$

$$\int_{\mathbb{R}^n} f(x)^p w(x)dx \leq C \int_{\mathbb{R}^n} g(x)^p w(x)dx.$$

By combination of these two Theorems we obtain

Theorem

Let $0 < \beta < \frac{n-1}{2}$, $n \geq 2$ *and* $\frac{2(n-1)}{n-1+2\beta} < p < \frac{2(n-1)}{n-1-2\beta}$. *Then for every* $w \in A_{\frac{4p\beta}{(2-p)(n-1)+2p\beta}}$

$$\int_{\mathbb{R}^n} T_\beta^* f(x)^p w^\gamma(x) dx \leq C \int_{\mathbb{R}^n} f(x)^p w^\gamma(x) dx,$$

where $\gamma = \frac{(2-p)(n-1)+2p\beta}{2(n-1)}$.

Theorem

Let $0 < \beta < \frac{n-1}{2}$, $n \geq 2$, $\frac{2(n-1)}{n-1+2\beta} < p < \frac{2(n-1)}{n-1-2\beta}$ *Suppouse that* $w \in A_1$ *such that*

$$\sum_{k=1}^{\infty} \frac{\varphi(x, 2^k r)|B(x, 2^k r)|^{\varepsilon}}{w(B(x, 2^k r))^{\gamma}} \leq C_1 \frac{\varphi(x, r)|B(x, r)|^{\varepsilon}}{w(B(x, r))^{\gamma}}$$

for every $x \in \mathbb{R}^n$ *and* $r > 0$, $\gamma = \frac{(2-p)(n-1)+2p\beta}{2(n-1)}$ *and for some* $\varepsilon > 0$. *Then operator* T_{β}^{*} *is bounded on* $\mathcal{M}_{p,\varphi}(w^{\gamma})$.

Theorem

Let $0 < \beta < \frac{n-1}{2}$, $n \geq 2$, $\frac{2(n-1)}{n-1+2\beta} < p < \frac{2(n-1)}{n-1-2\beta}$ *Suppouse that* $w \in A_1$ *such that*

$$\sum_{k=1}^{\infty} \frac{\varphi(x, 2^k r)|B(x, 2^k r)|^{\varepsilon}}{w(B(x, 2^k r))^{\gamma}} \leq C_1 \frac{\varphi(x, r)|B(x, r)|^{\varepsilon}}{w(B(x, r))^{\gamma}}$$

for every $x \in \mathbb{R}^n$ *and* $r > 0$, $\gamma = \frac{(2-p)(n-1)+2p\beta}{2(n-1)}$ *and for some* $\varepsilon > 0$. *Then operator* T^*_β *is bounded on* $\mathcal{M}_{p,\varphi}(w^\gamma)$.

Corollary

Let $0 < \beta < \frac{n-1}{2}$, $n \geq 2$, $\frac{2(n-1)}{n-1+2\beta} < p < \frac{2(n-1)}{n-1-2\beta}$, $0 < \kappa < \frac{(2-p)(n-1)+2\beta}{2(n-1)}$ *and* $w \in A_1$. *Then* T^*_β *is bounded on* $\mathcal{M}_{p,\kappa}(w^\gamma)$.

Given a Calderón- Zygmund singular integral operator T, and a function $b \in \mathrm{BMO}$, define the commutator $[b;T]$ to be the operator

$$[b;T]f(x) = b(x)Tf(x) - T(bf)(x).$$

These operators were shown to be bounded on $L_p(\mathbb{R}^n)$, $1 < p < \infty$, by Coifman, Rochberg and Weiss

R. Coifman, R. Rochberg, and G. Weiss, *Factorization theorems for Hardy spaces in several variables.* Ann. of Math., **103** (1976), 611–635.

it was shown in

C. Pérez, *Sharp estimates for commutators of singular integrals via iterations of the Hardy-Littlewood maximal function,* J. Fourier Anal. Appl., **3** (1997), 743–756.

that, for all $0 < p < \infty$ and all $w \in A_\infty$,

$$\int_{\mathbb{R}^n} |[b;T]f(x)|^p w(x)dx \le C \int_{\mathbb{R}^n} M^2 f(x)^p w(x)dx; \tag{11.1}$$

where $M^2 = M \circ M$. Hence, if $1 < p < \infty$ and $w \in A_p$, then $[b;T]$ is bounded on $L_p(w)$.

Theorem

Let $b \in \mathrm{BMO}$. *Suppose that* $w \in A_1$ *such that*

$$\sum_{k=1}^{\infty} \frac{\varphi(x,2^k r)|B(x,2^k r)|^{\varepsilon}}{w(B(x,2^k r))} \leq C_1 \frac{\varphi(x,r)|B(x,r)|^{\varepsilon}}{w(B(x,r))}$$

for every $x \in \mathbb{R}^n$ *and* $r > 0$, *and for some* $\varepsilon > 0$. *Then operator* $[b;T]$ *is bounded on* $\mathcal{M}_{p,\varphi}(w)$.

Theorem

Let $b \in \mathrm{BMO}$. Suppose that $w \in A_1$ such that

$$\sum_{k=1}^{\infty} \frac{\varphi(x, 2^k r)|B(x, 2^k r)|^{\varepsilon}}{w(B(x, 2^k r))} \le C_1 \frac{\varphi(x, r)|B(x, r)|^{\varepsilon}}{w(B(x, r))}$$

for every $x \in \mathbb{R}^n$ and $r > 0$, and for some $\varepsilon > 0$. Then operator $[b; T]$ is bounded on $\mathcal{M}_{p,\varphi}(w)$.

Corollary

Let $b \in \mathrm{BMO}$, $0 < \kappa < 1$ and $w \in A_1$. Then $[b; T]$ is bounded on $\mathcal{M}_{p,\kappa}(w)$.

The original inequality due to Coifman and Fefferman showed that Calderón-Zygmund singular integrals T could be controlled by the Hardy-Littlewood maximal operator: more precisely, given $w \in A_\infty$,

$$\int_{\mathbb{R}^n} |Tf(x)|^p w(x)dx \leq C \int_{\mathbb{R}^n} Mf(x)^p w(x)dx.$$

R. R. Coifman, *Distribution function inequalities for singular integrals,* Proc. Nat. Acad. Sci. U.S.A., **69** 1972, 2838-2839.

R. R. Coifman and C. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals,* Studia Math., **51** (1974), 241â250.

Theorem

Let $1 < p < \infty$. *Suppose that* $w \in A_1$ *such that*

$$\sum_{k=1}^{\infty} \frac{\varphi(x, 2^k r)|B(x, 2^k r)|^{\varepsilon}}{w(B(x, 2^k r))} \leq C_1 \frac{\varphi(x, r)|B(x, r)|^{\varepsilon}}{w(B(x, r))}$$

for every $x \in \mathbb{R}^n$ *and* $r > 0$, *and for some* $\varepsilon > 0$. *Then*

$$\|Tf\|_{\mathcal{M}_{p,\varphi}(w)} \leq C\|Mf\|_{\mathcal{M}_{p,\varphi}(w)}.$$

Theorem

Let $1 < p < \infty$*. Suppose that* $w \in A_1$ *such that*

$$\sum_{k=1}^{\infty} \frac{\varphi(x, 2^k r)|B(x, 2^k r)|^{\varepsilon}}{w(B(x, 2^k r))} \leq C_1 \frac{\varphi(x, r)|B(x, r)|^{\varepsilon}}{w(B(x, r))}$$

for every $x \in \mathbb{R}^n$ *and* $r > 0$*, and for some* $\varepsilon > 0$*. Then*

$$\|Tf\|_{\mathcal{M}_{p,\varphi}(w)} \leq C\|Mf\|_{\mathcal{M}_{p,\varphi}(w)}.$$

Corollary

Let $1 < p < \infty$*,* $0 < \kappa < 1$ *and* $w \in A_1$*. Then*

$$\|Tf\|_{\mathcal{M}_{p,\kappa}(w)} \leq C\|Mf\|_{\mathcal{M}_{p,\kappa}(w)}.$$

Given a locally integrable function f, the Hardy-Littlewood maximal function, Mf, is defined by

$$Mf(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| dy,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$.
Given a locally integrable function f the sharp maximal function $M^{\#}f$, is defined by

$$M^{\#}f(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B \left|f(y) - \frac{1}{|B|} \int_B f \right| dy,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$.
We have the weighted variant of the Fefferman and Stein inequality, for all p, $0 < p < \infty$, and $w \in A_\infty$,

$$\int_{\mathbb{R}^n} Mf(x)^p w(x) dx \le C \int_{\mathbb{R}^n} M^{\#}f(x)^p w(x) dx.$$

J. GARCÁ-CUERVA, AND J. L. RUBIO DE FRANCIA, Weighted Norm Inequalities and Related Topics. North-Holland Math. Stud. 116, North Holland, Amsterdam, (1985).

Theorem

Let $1 < p < \infty$*. Suppose that* $w \in A_1$ *such that*

$$\sum_{k=1}^{\infty} \frac{\varphi(x, 2^k r)|B(x, 2^k r)|^{\varepsilon}}{w(B(x, 2^k r))} \leq C_1 \frac{\varphi(x, r)|B(x, r)|^{\varepsilon}}{w(B(x, r))}$$

for every $x \in \mathbb{R}^n$ *and* $r > 0$*, and for some* $\varepsilon > 0$*. Then*

$$\|Mf\|_{\mathcal{M}_{p,\varphi}(w)} \leq C\|M^{\#} f\|_{\mathcal{M}_{p,\varphi}(w)}.$$

Theorem

Let $1 < p < \infty$*. Suppose that* $w \in A_1$ *such that*

$$\sum_{k=1}^{\infty} \frac{\varphi(x,2^k r)|B(x,2^k r)|^{\varepsilon}}{w(B(x,2^k r))} \le C_1 \frac{\varphi(x,r)|B(x,r)|^{\varepsilon}}{w(B(x,r))}$$

for every $x \in \mathbb{R}^n$ *and* $r > 0$*, and for some* $\varepsilon > 0$*. Then*

$$\|Mf\|_{\mathcal{M}_{p,\varphi}(w)} \le C\|M^{\#}f\|_{\mathcal{M}_{p,\varphi}(w)}.$$

Corollary

Let $1 < p < \infty$*,* $0 < \kappa < 1$ *and* $w \in A_1$*. Then*

$$\|Mf\|_{\mathcal{M}_{p,\kappa}(w)} \le C\|M^{\#}f\|_{\mathcal{M}_{p,\kappa}(w)}.$$

Thank you for your attention.