The extrapolation theorems for weighted generalized Morrey spaces

Amiran Gogatishvili

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(Weighted estimates)

The purpose of this talk is to present a method that allows us to obtain the boundedness of an operator

\[ T : \mathcal{M}_{p,\varphi}(w) \rightarrow \mathcal{M}_{p,\varphi}(w) \]

from the boundedness of

\[ T : L^p(w) \rightarrow L^p(w), \]

as soon as we have this information for a sufficiently large class of weights \( w \).
The Extrapolation theorem was first proved by Rubio de Francia


Recently the extrapolation method of Rubio de Francia was extended for Morrey space in the papers


J. Duoandikoetxea and M. Rosenthal Extension and boundedness of operators on Morrey spaces from extrapolation techniques and embeddings arXiv:1607.04565
The well-known Morrey spaces $\mathcal{M}_{p,\lambda}$ introduced by C. Morrey in 1938

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A weight is a locally integrable function on $\mathbb{R}^n$ which takes values in $(0, \infty)$ almost everywhere.

For a weight $w$ and a measurable set $E$, we denote $w(E) = \int_E w(x)dx$,

$|E|$ - is the Lebesgue measure of $E$.

$\chi_E$ - is characteristic function of $E$. $B(a, r)$ - is the open ball in $\mathbb{R}^n$ centred at $a$ with the radius $r$, $a \in \mathbb{R}^n$ and $r > 0$.

The weighted Lebesgue spaces with respect to the measure $w(x)dx$ are denoted by $L_{p,w}(\mathbb{R}^n)$ with $0 < p < \infty$, and

$$\|f\|_{p,w} = \left( \int_{\mathbb{R}^n} |f(x)|^p w(x)dx \right)^{\frac{1}{p}}.$$

we put $p' = \frac{p}{p-1}$, if $1 \leq p < \infty$. 
Definition

Let $1 \leq p < \infty$. Let be $\varphi$ a positive measurable function on $\mathbb{R}^n \times (0, \infty)$, non-decreasing with respect to $r$ for every $x \in \mathbb{R}^n$ and $w$ weighted function defined on $\mathbb{R}^n$. We define a weighted generalized Morrey space $\mathcal{M}_{p, \varphi}(w)$ by

$$\mathcal{M}_{p, \varphi}(w) := \left\{ f \in L_{p, w}^{\text{loc}}(\mathbb{R}^n) : \|f\|_{\mathcal{M}_{p, \varphi}(w)} < \infty \right\},$$

where

$$\|f\|_{\mathcal{M}_{p, \varphi}(w)} := \sup_{x \in \mathbb{R}^n, \ r > 0} \left( \frac{1}{\varphi(x, r)} \int_{B(x, r)} |f(y)|^p w(y) \, dy \right)^{\frac{1}{p}}.$$
If \( w \equiv 1 \), and \( \varphi(x, r) = \psi(r) \) for every \( x \in \mathbb{R}^n \), where \( \psi : (0, \infty) \to (0, \infty) \) is a non-decreasing function we obtain generalized Morey space \( \mathcal{M}_{p, \psi} \).
If $w \equiv 1$, and $\varphi(x, r) = \psi(r)$ for every $x \in \mathbb{R}^n$, where $\psi : (0, \infty) \to (0, \infty)$ is a non-decreasing function we obtain generalized Morey space $\mathcal{M}_{p, \psi}$

If $\psi(r) = r^\lambda$, $0 < \lambda < n$, then $\mathcal{M}_{p, \varphi}(w)$ becomes the classical Morrey space $\mathcal{M}_{p, \lambda}$. 
If \( w \equiv 1 \), and \( \phi(x, r) = \psi(r) \) for every \( x \in \mathbb{R}^n \), where \( \psi : (0, \infty) \to (0, \infty) \) is a non-decreasing function we obtain generalized Morrey space \( M_{p, \psi} \).

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If \( \phi(x, r) = w(B(x, r))^{\kappa} \), \( 0 < \kappa < 1 \) we get weighted Morrey spaces \( M_{p, \kappa}(w) \), consider in

If $w \equiv 1$, and $\varphi(x, r) = \psi(r)$ for every $x \in \mathbb{R}^n$, where $\psi : (0, \infty) \rightarrow (0, \infty)$ is a non-decreasing function we obtain generalized Morey space $\mathcal{M}_{p, \psi}$.

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For more properties about the weighted generalized Morrey space we refer the reader to

Muckenhoupt $A_p$ weights, $w \in A_p, 1 \leq p \leq \infty$, if there exists a constant $C$ such that for every balls $B \subset \mathbb{R}^n$ we have

$$\frac{1}{|B|} \int_B w(x) dx \left( \frac{1}{|B|} \int_B w(x)^{1-p'} dx \right)^{p-1} \leq C,$$

when $1 < p < \infty$, and for $p = 1$

$$\frac{1}{|B|} \int_B w(x) dx \leq C \text{ess inf}_{x \in B} w(x).$$

The class $A_\infty$ is defined as $A_\infty = \cup_{p \geq 1} A_p$. 

We will consider following weighted norm inequalities

\[ \int_{\mathbb{R}^n} Tf(x)^{p_0} w^{\delta}(x) dx \leq C_0 \int_{\mathbb{R}^n} f(x)^{p_0} w^{\delta}(x) dx, \quad (1.1) \]

where \( T \) is some operator and \( w \in A_1, 0 < p_0 < \infty, 0 < \delta \leq 1. \) (In other words, \( T \) is defined and bounded on \( L^{p_0}(w^{\delta}) \).) The constant \( C_0 \) is assumed to depend only on the \( A_1 \) constant of \( w. \)
Hereafter $\mathcal{F}$ will denote a family of pairs $(f, g)$ of non-negative, measurable functions on $\mathbb{R}^n$. We say that an inequality

$$\int_{\mathbb{R}^n} f(x)^p w(x) \, dx \leq C \int_{\mathbb{R}^n} g(x)^p w(x) \, dx$$

holds for any $(f, g) \in \mathcal{F}$ and $w \in A_q$ (for some $q$, $1 \leq q < \infty$), we mean that it holds for any pair in $\mathcal{F}$ such that the left-hand side is finite, and the constant $C$ depends only $p$ and the $A_q$ constant of $w$. 
Theorem

Given a family $\mathcal{F}$, suppose that for some $p, \delta, 0 < p < \infty$, $0 < \delta \leq 1$, and for every weight $w \in A_1$

$$
\int_{\mathbb{R}^n} f(x)^p w^\delta(x) dx \leq C_0 \int_{\mathbb{R}^n} g(x)^p w^\delta(x) dx, \quad (f, g) \in \mathcal{F}.
$$

(2.1)

Let $w \in A_1$ such that

$$
\sum_{k=1}^{\infty} \frac{\varphi(x, 2^k r)|B(x, 2^k r)|^{\varepsilon_0}}{w(B(x, 2^k r))^\delta} \leq C_1 \frac{\varphi(x, r)|B(x, r)|^{\varepsilon_0}}{w(B(x, r))^\delta}
$$

(2.2)

for every $x \in \mathbb{R}^n$ and $r > 0$, and for some $\varepsilon_0 > 0$. Then for all $(f, g) \in \mathcal{F}$

$$
\|f\|_{\mathcal{M}_{p, \varphi}(w^\delta)} \leq C_2 \|g\|_{\mathcal{M}_{p, \varphi}(w^\delta)}.
$$

(2.3)
Theorem

Given a family $\mathcal{F}$, assume that for some $p, q$ and $\delta$, $0 < p \leq q < \infty$, $0 < \delta \leq 1$ and every weight $w \in A_1$,

$$\left( \int_{\mathbb{R}^n} f(x)^q w(x)^{\delta} \, dx \right)^{1/q} \leq C_0 \left( \int_{\mathbb{R}^n} g(x)^p w(x)^{\delta p / q} \, dx \right)^{1/p} \quad , \quad (f, g) \in \mathcal{F}. \quad (2.4)$$

Let $w \in A_1$ such that

$$\sum_{k=1}^{\infty} \frac{\varphi(x, 2^k r)^{p/q} |B(x, 2^k r)|^{\varepsilon_0}}{w(B(x, 2^k r))} \leq C_1 \frac{\varphi(x, r)^{p/q} |B(x, r)|^{\varepsilon_0}}{w(B(x, r))^{\delta p / q}} \quad (2.5)$$

for every $x \in \mathbb{R}^n$ and $r > 0$, and for some $\varepsilon_0 > 0$. Then for all $(f, g) \in \mathcal{F}$

$$\|f\|_{\mathcal{M}_{q, \varphi}(w^{\delta})} \leq C_2 \|g\|_{\mathcal{M}_{p, \varphi^{p/q}}(w^{\delta / q})} \quad (2.6)$$
Uusing rescaling argument we obtain following

**Corollary**

Suppose that for some $p_0$, $0 < p_0 < \infty$, the family $\mathcal{F}$ is such that for all $w \in A_1$

$$\int_{\mathbb{R}^n} f(x)^{p_0} w(x) \, dx \leq C_0 \int_{\mathbb{R}^n} g(x)^{p_0} w(x) \, dx, \quad (f, g) \in \mathcal{F}. \quad (2.7)$$

Given $\varphi$ and $w \in A_1$, such that

$$\sum_{k=1}^{\infty} \frac{\varphi(x, 2^k r)|B(x, 2^k r)|^{\varepsilon_0}}{w(B(x, 2^k r))} \leq C_1 \frac{\varphi(x, r)|B(x, r)|^{\varepsilon_0}}{w(B(x, r))} \quad (2.8)$$

for every $x \in \mathbb{R}^n$ and $r > 0$, and for some $\varepsilon_0 > 0$. Suppose $p_0 < p < \infty$. Then for all $(f, g) \in \mathcal{F}$

$$\|f\|_{\mathcal{M}_{p, \varphi}(w)} \leq C_2 \|g\|_{\mathcal{M}_{p, \varphi}(w)}. \quad (2.9)$$
Theorem

Assume that $0 < p \leq q < \infty$, $0 < \delta \leq 1$. Let $w \in A_1$ such that

$$\sum_{k=1}^{\infty} \frac{\varphi(x, 2^k r)^{\frac{p}{\delta p}} |B(x, 2^k r)|^{\varepsilon_0}}{w(B(x, 2^k r)))^{\frac{\delta p}{\delta q}}} \leq C_0 \frac{\varphi(x, r)^{\frac{p}{\delta p}} |B(x, r)|^{\varepsilon_0}}{w(B(x, r)))^{\frac{\delta p}{\delta q}}}$$

for every $x \in \mathbb{R}^n$ and $r > 0$, and for some $\varepsilon_0 > 0$. Then there exists a constant $C_1 > 0$ and $s > 1$ such that for all $f \in \mathcal{M}_{p, \varphi^{\frac{\delta p}{\delta q}}}$, and all ball $B(x_0, r)$ holds

$$\left( \int_{\mathbb{R}^n} |f(x)|^p (M(\chi_{B(x_0, r)} w^s)(x))^{\frac{\delta p}{sq}} dx \right)^{\frac{1}{p}} \leq C_1 \varphi(x_0, r)^{\frac{1}{q}} \|f\|_{\mathcal{M}_{p, \varphi^{\frac{\delta p}{\delta q}}}}. \quad (2.10)$$
If we take $p = q$ we get following result

**Theorem**

Assume that $0 < p < \infty$, $0 < \delta \leq 1$. Let $w \in A_1$ such that

$$\sum_{k=1}^{\infty} \frac{\varphi(x, 2^k r) |B(x, 2^k r)|^{\varepsilon_0}}{w(B(x, 2^k r))^\delta} \leq C_1 \frac{\varphi(x, r) |B(x, r)|^{\varepsilon_0}}{w(B(x, r))^\delta}$$

for every $x \in \mathbb{R}^n$ and $r > 0$, and for some $\varepsilon_0 > 0$. Then there exists a constant $C_1 > 0$ and $s > 1$ such that for all $f \in \mathcal{M}_{p, \varphi}(w^\delta)$, and all ball $B(x_0, r)$ holds

$$\left( \int_{\mathbb{R}^n} |f(x)|^p \left( M(\chi_{B(x_0, r)} w^s)(x) \right)^{\delta s} dx \right)^{\frac{1}{p}} \leq C_1 \varphi(x_0, r)^{\frac{1}{p}} \| f \|_{\mathcal{M}_{p, \varphi}(w^\delta)}.$$
If we take \( p = q \) we get following result

**Theorem**

Assume that \( 0 < p < \infty \), \( 0 < \delta \leq 1 \). Let \( w \in A_1 \) such that

\[
\sum_{k=1}^{\infty} \frac{\varphi(x, 2^k r) |B(x, 2^k r)|^{\varepsilon_0}}{w(B(x, 2^k r))^{\delta}} \leq C_1 \frac{\varphi(x, r) |B(x, r)|^{\varepsilon_0}}{w(B(x, r))^{\delta}}
\]

for every \( x \in \mathbb{R}^n \) and \( r > 0 \), and for some \( \varepsilon_0 > 0 \). Then there exists a constant \( C_1 > 0 \) and \( s > 1 \) such that for all \( f \in M_{p, \varphi}(w^\delta) \), and all ball \( B(x_0, r) \) holds

\[
\left( \int_{\mathbb{R}^n} |f(x)|^p (M(\chi_{B(x_0, r)} w^s)(x))^{\frac{\delta}{s}} \, dx \right)^{\frac{1}{p}} \leq C_1 \varphi(x_0, r)^{\frac{1}{p}} \|f\|_{M_{p, \varphi}(w^\delta)}.
\]

Therefore, we have the inequalities of the form (1.1) with weights

\( v(x) = (M(\chi_{B(x_0, r)} w^s)(x))^{\frac{1}{s}} \) for some \( s > 1 \). As the family \( \mathcal{F} \) in the hypothesis and conclusion of Theorem is the same, so the goal is to find a large, reasonable family \( \mathcal{F} \) such that (1.1) holds with a constant depending only on \( p_0 \) and the \( A_1 \) constant of \( w \).
Theorem

Given a operator $T$, suppose that for some $p$, $q$ and $\delta$ $0 < p \leq q < \infty$, $0 < \delta \leq 1$ and every $w \in A_1$, there exists constant $C_0$ depending on $A_1$ constant of $w$ such that

$$
\left( \int_{\mathbb{R}^n} |Tf(x)|^q w(x)^{\delta} \, dx \right)^{\frac{1}{q}} \leq C_0 \left( \int_{\mathbb{R}^n} |f(x)|^p w(x)^{\frac{\delta p}{q}} \, dx \right)^{\frac{1}{p}}, \quad f \in L_{p,w}^{\frac{\delta p}{q}}.
$$

Given $\varphi$ and $w \in A_1$, such that

$$
\sum_{k=1}^{\infty} \varphi(x,2^k r)^{\frac{p}{q}} w(B(x,2^k r))^{\frac{\delta p}{q}} \leq C_1 \varphi(x,r)^{\frac{p}{q}} w(B(x,r))^{\frac{\delta p}{q}} \quad (2.11)
$$

for every $x \in \mathbb{R}^n$ and $r > 0$, and for some $\varepsilon_0 > 0$. Then there exists constant $C_1$ such that for all $f \in \mathcal{M}_{p,\varphi}^{\frac{\delta p}{q}}(w^{\frac{\delta p}{q}})$

$$
\|Tf\|_{\mathcal{M}_{q,\varphi}^{\delta}(w^{\delta})} \leq C_2 \|f\|_{\mathcal{M}_{p,\varphi}^{\frac{\delta p}{q}}(w^{\frac{\delta p}{q}})}. \quad (2.12)
$$
**Theorem**

Given a operators $T, S$ suppose that for some $p, q$, $\delta$, $0 < p \leq q < \infty$, $0 < \delta \leq 1$ and every $w \in A_1$, there exists constant $C_0$ depending on $A_1$ constant of $w$ such that

$$
\left( \int_{\mathbb{R}^n} |Tf(x)|^q w(x)^\delta \, dx \right)^{1/q} \leq C_0 \left( \int_{\mathbb{R}^n} |Sf(x)|^p w(x)^{\frac{\delta p}{q}} \, dx \right)^{1/p}, \quad Sf \in L^p_{p,w} \frac{\delta p}{q}.
$$

Given $\varphi$ and $w \in A_1$, such that

$$
\sum_{k=1}^{\infty} \frac{\varphi(x, 2^k r)^{\frac{p}{q}} |B(x, 2^k r)|^{\varepsilon_0}}{w(B(x, 2^k r))^{\frac{\delta p}{q}}} \frac{w(B(x, 2^k r))^{\frac{\delta p}{q}}}{w(B(x, r))^{\frac{\delta p}{q}}} \leq C_1 \frac{\varphi(x, r)^{\frac{p}{q}} |B(x, r)|^{\varepsilon_0}}{w(B(x, r))^{\frac{\delta p}{q}}}
$$

(2.13)

for every $x \in \mathbb{R}^n$ and $r > 0$, and for some $\varepsilon_0 > 0$. Then there exists constant $C_1$ such that for all $Sf \in \mathcal{M}_{p,\varphi}^\frac{\delta p}{q} (w^\frac{p}{q})$

$$
\|Tf\|_{\mathcal{M}_{q,\varphi}(w^\delta)} \leq C_2 \|Sf\|_{\mathcal{M}_{p,\varphi}^\frac{\delta p}{q} (w^\frac{p}{q})}.
$$

(2.14)
Theorem

Given a operator $T$, $S$ suppose that for some $p$, $\delta$, $0 < p < \infty$, $0 < \delta \leq 1$ and every $w \in A_1$, there exists constant $C_0$ depending on $A_1$ constant of $w$ such that

$$\int_{\mathbb{R}^n} |Tf(x)|^p w(x)^\delta \, dx \leq C_0 \int_{\mathbb{R}^n} |Sf(x)|^p w(x)^\delta \, dx, \quad Sf \in L_{p,w^\delta}.$$  

Given $\varphi$ and $w \in A_1$, such that

$$\sum_{k=1}^{\infty} \frac{\varphi(x, 2^k r)|B(x, 2^k r)|^{\varepsilon_0}}{w(B(x, 2^k r))^\delta} \leq C_1 \frac{\varphi(x, r)|B(x, r)|^{\varepsilon_0}}{w(B(x, r))^\delta} \quad (2.15)$$

for every $x \in \mathbb{R}^n$ and $r > 0$, and for some $\varepsilon_0 > 0$. Then there exists constant $C_1$ such that for all $Sf \in \mathcal{M}_{p,\varphi}(w^\delta)$

$$\|Tf\|_{\mathcal{M}_{p,\varphi}(w^\delta)} \leq C_2 \|Sf\|_{\mathcal{M}_{p,\varphi}(w^\delta)}.$$
Define the Spherical Maximal operator $\mathcal{M}$, by

$$\mathcal{M}f(x) := \sup_{t>0} |\mu_t \ast f(x)|,$$

where $\mu_t$ denotes the normalized surface measure on the sphere of center 0 and radius $t$ in $\mathbb{R}^n$. 

Theorem

If $n > 2$, $n - \frac{2}{\sigma} < \infty$ and $w$ belongs to the $A_1$ Muckenhoupt class, then

$$\|\mathcal{M}f\|_{\sigma, w} \lesssim C \|f\|_{\sigma, w},$$

where $\delta = n - \frac{2}{\sigma}$.

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**Theorem**

If $n > 2$, $\frac{n}{n-1} < \sigma < \infty$ and $w$ belongs to the $A_1$ Muckenhoupt class, then

$$
\|\mathcal{M}f\|_{\sigma, w^\delta} \lesssim C \|f\|_{\sigma, w^\delta},
$$

where $\delta = \frac{n-2}{n-1}$.

Theorem

Let \( n > 2, \ \frac{n}{n-1} < p < \infty \) and \( \delta = \frac{n-2}{n-1} \). Let be \( w \in A_1 \) such that

\[
\sum_{k=1}^{\infty} \frac{\varphi(x, 2^k r)|B(x, 2^k r)|^{\varepsilon}}{w(B(x, 2^k r))^{\delta}} \leq C_1 \frac{\varphi(x, r)|B(x, r)|^{\varepsilon}}{w(B(x, r))^{\delta}}
\]

(4.2)

for every \( x \in \mathbb{R}^n \) and \( r > 0 \), and for some \( \varepsilon > 0 \). Then

\[
\|Mf\|_{M_{p, \varphi}(w^{\delta})} \leq C\|f\|_{M_{p, \varphi}(w^{\delta})}.
\]

(4.3)
Corollary

Let $n > 2$, $\frac{n}{n-1} < p < \infty$, $0 < \kappa < \frac{n-2}{n-1}$, $\delta = \frac{n-2}{n-1}$, and $w \in A_1$. Then

$$\|Mf\|_{\mathcal{M}_{p,\kappa}(w^\delta)} \leq C\|f\|_{\mathcal{M}_{p,\kappa}(w^\delta)}. \quad (4.4)$$
Corollary

Let $n > 2$, $\frac{n}{n-1} < p < \infty$, $0 < \kappa < \frac{n-2}{n-1}$, $\delta = \frac{n-2}{n-1}$, and $w \in A_1$. Then

$$\| Mf\|_{M_{p,\kappa}(w^{\delta})} \leq C\| f\|_{M_{p,\kappa}(w^{\delta})}. \quad (4.4)$$

If $w \equiv 1$ we get

Corollary

Let $n > 2$, $\frac{n}{n-1} < p < \infty$, $0 < \lambda < n\frac{n-2}{n-1}$. Then

$$\| Mf\|_{M_{p,\lambda}} \leq C\| f\|_{M_{p,\lambda}}. \quad (4.5)$$
Denote by $\mu_t$ the normalized surface measure on the sphere in $\mathbb{R}^n$ with center 0 and radius $t$. The maximal operator related to spherical means is given by

$$M^\alpha = \sup_{t>0} |t^\alpha \mu_t * f|.$$
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The weighted $L^p \to L^q$ estimate for the maximal operators $\mathcal{M}^\alpha$, was investigated in the paper


**Theorem**

Suppose that $n > 2$, $\frac{n}{n-1} < p < q < n$, that $\alpha = \frac{n}{p} - \frac{n}{q}$, and that \(\max\{0, 1 - q/p'\} < \gamma \leq 1 - q/n\). Suppose also that $w$ is in $A_s$, where

$$s = \frac{q + 2p' \gamma - p'}{p' \gamma}.$$

Then there exists a constant $C$ such that

$$\|\mathcal{M}^\alpha f\|_{q,w^\gamma} \leq C \|f\|_{p,w^{\gamma p/q}}.$$
Theorem

Let $n > 2$, $\frac{n}{n-1} < p < q < n$, $\alpha = n/p - n/q$, and
\[ \max\{0, 1 - q/p'\} < \gamma \leq 1 - q/n. \]
Suppose also that $w$ is in $A_1$ for which

\[
\sum_{k=1}^{\infty} \frac{\varphi(x, 2^k r)^{p \gamma q}}{w(B(x, 2^k r))^{\gamma q}} \leq C_1 \frac{\varphi(x, r)^{p \gamma q}}{w(B(x, r))^{\gamma q}}
\]

(7.1)

for every $x \in \mathbb{R}^n$ and $r > 0$, and for some $\varepsilon > 0$. Then

\[
\|\mathcal{M}^\alpha f\|_{\mathcal{M}_{q, \varphi}(w^\gamma)} \leq C \|f\|_{\mathcal{M}_{p, \varphi/p/q}(w^{\gamma q})}.
\]

(7.2)
Corollary

Let $n > 2$, $\frac{n}{n-1} < p < q < n$, $\alpha = n/p - n/q$, $\max\{0, 1 - q/p'\} < \gamma \leq 1 - q/n$, $0 < \kappa < \gamma$. Suppose that $w \in A_1$. Then

$$\|M^\alpha f\|_{M_{q,\kappa}(w^\gamma)} \leq C\|f\|_{M_{p,\kappa}(w^{p\gamma}/q)}.$$
Corollary

Let $n > 2$, $\frac{n}{n-1} < p < q < n$, $\alpha = \frac{n}{p} - \frac{n}{q}$, $\max\{0, 1 - \frac{q}{p'}\} < \gamma \leq 1 - \frac{q}{n}$, $0 < \kappa < \gamma$. Suppose that $w \in A_1$. Then

$$\|\mathcal{M}^\alpha f\|_{\mathcal{M}_{q, \kappa}(w^\gamma)} \leq C \|f\|_{\mathcal{M}_{p, \kappa}(w^{p\gamma/q})}.$$  

Corollary

Let $n > 2$, $\frac{n}{n-1} < p < q < n$, $\alpha = \frac{n}{p} - \frac{n}{q}$, $0 < \lambda < n - q$. Then

$$\|\mathcal{M}^\alpha f\|_{\mathcal{M}_{q, \lambda}} \leq C \|f\|_{\mathcal{M}_{p, \lambda p/q}}.$$
Theorem (D.K. Watson)

Let $n \geq 2$, $1 < r < \infty$ and let $Tf(x) = p.v.K \ast f(x)$ be singular integral operator with "rough" kernel

$$K(x) = h(|x|) \frac{\Omega(x)}{|x|^n},$$

where $\Omega$ is homogeneous of degree 0 on $\mathbb{R}^n$, $\Omega \in L^r(S^{n-1})$, where $S^{n-1}$ denote the unit sphere in $\mathbb{R}^n$. $\Omega$ has average 0 on $S^{n-1}$, and $h$ is a measurable function on $(0, \infty)$ satisfying

$$\int_R^{2R} |h(t)|^r dt \leq CR \quad \text{for all } R > 0.$$

Then $T$ is bounded on $L_{p,w}(\mathbb{R}^n)$,

$$\|Tf\|_{p,w} \leq C\|f\|_{p,w},$$

in each of the following situations:
(A) if $r' \leq p < \infty$, and $w \in A_{\frac{p}{r'}}$, or
(B) if $1 < p \leq r$, $p \neq \infty$ and $w^\frac{1}{p-1} \in A_{\frac{p'}{r}}$, or
(C) if $1 < p < \infty$ and $w'^r \in A_p$. 
Theorem

Let $1 < r < \infty$ and $r' \leq p < \infty$. Suppose that $w$ is in $A_1$ for which

$$\sum_{k=1}^{\infty} \frac{\varphi(x, 2^k t) |B(x, 2^k t)|^\varepsilon}{w(B(x, 2^k t))} \leq C_1 \frac{\varphi(x, t) |B(x, t)|^\varepsilon}{w(B(x, t))}$$

(8.1)

for every $x \in \mathbb{R}^n$ and $t > 0$, and for some $\varepsilon > 0$. Then

$$\| Tf \|_{\mathcal{M}_p, \varphi(w)} \leq C \| f \|_{\mathcal{M}_p, \varphi(w)}.$$

(8.2)
**Theorem**

Let $1 < r < \infty$ and $r' \leq p < \infty$. Suppose that $w$ is in $A_1$ for which

$$
\sum_{k=1}^{\infty} \frac{\varphi(x, 2^k t)|B(x, 2^k t)|^\varepsilon}{w(B(x, 2^k t))} \leq C_1 \frac{\varphi(x, t)|B(x, t)|^\varepsilon}{w(B(x, t))} 
$$

(8.1)

for every $x \in \mathbb{R}^n$ and $t > 0$, and for some $\varepsilon > 0$. Then

$$
\|Tf\|_{M_p, \varphi(w)} \leq C \|f\|_{M_p, \varphi(w)}. 
$$

(8.2)

**Corollary**

Let $1 < r < \infty$ and $r' \leq p < \infty$. Suppose also that $w$ is in $A_1$ and $0 < \kappa < 1$. Then

$$
\|Tf\|_{M_p, \kappa(w)} \leq C \|f\|_{M_p, \kappa(w)}. 
$$
Theorem

Let $1 < r < \infty$ and $p < r$. Suppose also that $w$ is in $A_1$ for which

$$\sum_{k=1}^{\infty} \frac{\varphi(x, 2^k t) |B(x, 2^k t)|^\varepsilon}{w(B(x, 2^k t))} \leq C_1 \frac{\varphi(x, t) |B(x, t)|^\varepsilon}{w(B(x, t))}$$

for every $x \in \mathbb{R}^n$ and $t > 0$, and for some $\varepsilon > 0$. Then

$$\|Tf\|_{\mathcal{M}_{r-p}(w^{-r/p})} \leq C \|f\|_{\mathcal{M}_{r-p}(w^{-r/p})}.$$
Theorem

Let $1 < r < \infty$ and $p < r$. Suppose also that $w$ is in $A_1$ for which

$$
\sum_{k=1}^{\infty} \frac{\varphi(x, 2^k t)|B(x, 2^k t)|^\varepsilon}{w(B(x, 2^k t))^{\frac{r-p}{r}}} \leq C_1 \frac{\varphi(x, t)|B(x, t)|^\varepsilon}{w(B(x, t))^{\frac{r-p}{r}}}
$$

(8.3)

for every $x \in \mathbb{R}^n$ and $t > 0$, and for some $\varepsilon > 0$. Then

$$
\| Tf \|_{\mathcal{M}_{p, \varphi}(w^{\frac{r-p}{r}})} \leq C \| f \|_{\mathcal{M}_{p, \varphi}(w^{\frac{r-p}{r}})}.
$$

(8.4)

Corollary

Let $1 < r < \infty$ and $p < r$. Suppose that $w$ is in $A_1$ and $0 < \kappa < \frac{r-p}{r}$. Then

$$
\| Tf \|_{\mathcal{M}_{p, \kappa}(w^{\frac{r-p}{r}})} \leq C \| f \|_{\mathcal{M}_{p, \kappa}(w^{\frac{r-p}{r}})}.
$$
**Theorem**

Let $1 < r, p < \infty$. Suppose that $w$ is in $A_1$ for which

$$\sum_{k=1}^{\infty} \frac{\varphi(x, 2^k t) |B(x, 2^k t)|^\varepsilon}{w(B(x, 2^k t))^{\frac{r-1}{r}}} \leq C_1 \frac{\varphi(x, t) |B(x, t)|^\varepsilon}{w(B(x, t))^{\frac{r-1}{r}}}$$  \hspace{1cm} (8.5)

for every $x \in \mathbb{R}^n$ and $t > 0$, and for some $\varepsilon > 0$. Then

$$\|Tf\|_{\mathcal{M}_{p, \varphi}(w^{\frac{r-1}{r}})} \leq C \|f\|_{\mathcal{M}_{p, \varphi}(w^{\frac{r-1}{r}})}.$$  \hspace{1cm} (8.6)
Theorem

Let $1 < r, p < \infty$. Suppose that $w$ is in $A_1$ for which

$$
\sum_{k=1}^{\infty} \frac{\varphi(x, 2^k t)|B(x, 2^k t)|^\varepsilon}{w(B(x, 2^k t))^{r-1 \over r}} \leq C_1 \frac{\varphi(x, t)|B(x, t)|^\varepsilon}{w(B(x, t))^{r-1 \over r}}
$$

(8.5)

for every $x \in \mathbb{R}^n$ and $t > 0$, and for some $\varepsilon > 0$. Then

$$
\|Tf\|_{\mathcal{M}_{p, \varphi}(w^{r-1 \over r})} \leq C \|f\|_{\mathcal{M}_{p, \varphi}(w^{r-1 \over r})}.
$$

(8.6)

Corollary

Let $1 < r, p < \infty$. Suppose that $w$ is in $A_1$ and $0 < \kappa < {r-1 \over r}$. Then

$$
\|Tf\|_{\mathcal{M}_{p, \kappa}(w^{r-1 \over r})} \leq C \|f\|_{\mathcal{M}_{p, \kappa}(w^{r-1 \over r})}.
$$
Let $\theta(\xi)$ be a smooth radial cut-off function $\theta(\xi) = 1$ if $|\xi| \geq 1$ and $\theta(\xi) = 0$ if $|\xi| \leq 1/2$. We will consider the multipliers

$$\hat{T}_{b,a}f(\xi) = \theta(\xi) \frac{e^{i|\xi|^b}}{|\xi|^a} \hat{f}(\xi),$$

where $0 < b < 1$ and $0 < a < nb/2$. C. Fefferman proved that if $0 < a < nb/2$, and $p$ is such that $|1/p - 1/2| \leq a/nb$, then

$$\|T_{b,a}\|_p \leq c_p \|f\|_p.$$

The weighted extension of Fefferman’s theorem was obtained by Chanillo,

**Theorem**

Let $1 < p < \infty$, $\alpha = nb|1/p - 1/2|$, and $w \in A_p$. Then for $\alpha \leq a \leq nb/2$, and for $\gamma$, such that $\gamma = (a - \alpha)/(nb/2 - \alpha)$ we have

\[
\|T_{b,a}f\|_{p,w^\gamma} \leq C_p \|f\|_{p,w^\gamma}.
\] (9.1)

Theorem

Let for some $1 < p < \infty$, $\alpha = nb|1/p - 1/2|$, $\alpha < a \leq nb/2$. Let
$\gamma = (a - \alpha)/(nb/2 - \alpha)$ and

$$
\sum_{k=1}^{\infty} \frac{\varphi(x, 2^k r)|B(x, 2^k r)|^\varepsilon}{w(B(x, 2^k r))^\gamma} \leq C_1 \frac{\varphi(x, r)|B(x, r)|^\varepsilon}{w(B(x, r))^\gamma}
$$

for every $x \in \mathbb{R}^n$ and $r > 0$, and for some $\varepsilon > 0$. Then operator $T_{b,a}$ is bounded on $\mathcal{M}_{p,\varphi}(w^\gamma)$. 

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Corollary

Let for some $1 < p < \infty$, $\alpha = nb|1/p - 1/2|$, $\alpha < a \leq nb/2$. Let $\gamma = (a - \alpha)/(nb/2 - \alpha)$ and $0 < \kappa < \gamma$. Then operator $T_{b,a}$ is bounded on $\mathcal{M}_{p,\kappa}(w^\gamma)$.

Corollary

Let for some $1 < p < \infty$, $\alpha = nb|1/p - 1/2|$, $\alpha < a \leq nb/2$, and $0 < \lambda < n(a - \alpha)/(nb/2 - \alpha)$. Then operator $T_{b,a}$ is bounded on $\mathcal{M}_{p,\lambda}$. 
The Bochner-Riesz operator in $\mathbb{R}^n$, $(n \geq 2)$ are defined for $\beta > 0$, as

$$\widehat{T^r_\beta}(\xi) = \left(1 - \frac{|\xi|^2}{r^2}\right)\hat{f}(\xi)$$

with $t_+ = \max(t, 0)$, and the maximal Bochner-Riesz operator is defined by

$$T^*_\beta f(x) = \sup_{r>0} |T^r_\beta f(x)|.$$

**Theorem**

*If* $0 < \beta < \frac{n-1}{2}$, *then* $T^*_\beta$ *is bounded on* $L^2(w^{\frac{2\beta}{n-1}})$ *for* $w \in A_2$.

---

The following theorem in case $p = 2$ was proved by Duoandikoetxea, et. al. in


For the case $p \neq 2$

**Theorem**

*Given* $\delta, \ 0 < \delta < 1$, *suppose that for all* $w \in A_2$

$$
\int_{\mathbb{R}^n} f(x)^2 w^\delta(x) \, dx \leq C \int_{\mathbb{R}^n} g(x)^2 w^\delta(x) \, dx, \quad (f, g) \in \mathcal{F}.
$$

*Then for all* $p, \ \frac{2}{1+\delta} < p < \frac{2}{1-\delta}$, *and every* $w^{\frac{2}{2-p(1-\delta)}} \in A_2^{\frac{2p\delta}{2-p(1-\delta)}}$

$$
\int_{\mathbb{R}^n} f(x)^p w(x) \, dx \leq C \int_{\mathbb{R}^n} g(x)^p w(x) \, dx.
$$
By combination of these two Theorems we obtain

**Theorem**

Let $0 < \beta < \frac{n-1}{2}$, $n \geq 2$ and $\frac{2(n-1)}{n-1+2\beta} < p < \frac{2(n-1)}{n-1-2\beta}$. Then for every $w \in A_{\frac{4p\beta}{(2-p)(n-1)+2p\beta}}$

$$
\int_{\mathbb{R}^n} T_{\beta}^* f(x)^p w^\gamma(x) \, dx \leq C \int_{\mathbb{R}^n} f(x)^p w^\gamma(x) \, dx,
$$

where $\gamma = \frac{(2-p)(n-1)+2p\beta}{2(n-1)}$. 
Theorem

Let $0 < \beta < \frac{n-1}{2}$, $n \geq 2$, \( \frac{2(n-1)}{n-1+2\beta} < p < \frac{2(n-1)}{n-1-2\beta} \) Suppose that $w \in A_1$ such that

$$\sum_{k=1}^{\infty} \varphi(x, 2^k r)|B(x, 2^k r)|^\varepsilon \leq C_1 \varphi(x, r)|B(x, r)|^\varepsilon$$

$$w(B(x, 2^k r))^\gamma \leq C_1 w(B(x, r))^\gamma$$

for every $x \in \mathbb{R}^n$ and $r > 0$, $\gamma = \frac{(2-p)(n-1)+2p\beta}{2(n-1)}$ and for some $\varepsilon > 0$. Then operator $T^*_\beta$ is bounded on $\mathcal{M}_{p, \varphi}(w^\gamma)$. 

Corollary

Let $0 < \beta < \frac{n-1}{2}$, $n \geq 2$, \( \frac{2(n-1)}{n-1+2\beta} < p < \frac{2(n-1)}{n-1-2\beta} \), $0 < \kappa < \frac{(2-p)(n-1)+2p\beta}{2(n-1)}$ and $w \in A_1$. Then $T^*_\beta$ is bounded on $\mathcal{M}_{p, \kappa}(w^\gamma)$. 

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The extrapolation theorems for weighted generalized Morrey spaces
**Theorem**

Let $0 < \beta < \frac{n-1}{2}$, $n \geq 2$, $\frac{2(n-1)}{n-1+2\beta} < p < \frac{2(n-1)}{n-1-2\beta}$ Suppouse that $w \in A_1$ such that

$$\sum_{k=1}^{\infty} \frac{\varphi(x, 2^k r)|B(x, 2^kr)|^\varepsilon}{w(B(x, 2^k r))^{\gamma}} \leq C_1 \frac{\varphi(x, r)|B(x, r)|^\varepsilon}{w(B(x, r))^{\gamma}}$$

for every $x \in \mathbb{R}^n$ and $r > 0$, $\gamma = \frac{(2-p)(n-1)+2p\beta}{2(n-1)}$ and for some $\varepsilon > 0$. Then operator $T_{\beta}^*$ is bounded on $\mathcal{M}_{p,\varphi}(w^{\gamma})$.

**Corollary**

Let $0 < \beta < \frac{n-1}{2}$, $n \geq 2$, $\frac{2(n-1)}{n-1+2\beta} < p < \frac{2(n-1)}{n-1-2\beta}$, $0 < \kappa < \frac{(2-p)(n-1)+2\beta}{2(n-1)}$ and $w \in A_1$. Then $T_{\beta}^*$ is bounded on $\mathcal{M}_{p,\kappa}(w^{\gamma})$. 
Given a Calderón- Zygmund singular integral operator $T$, and a function $b \in \text{BMO}$, define the commutator $[b; T]$ to be the operator

$$[b; T]f(x) = b(x)Tf(x) - T(bf)(x).$$

These operators were shown to be bounded on $L_p(\mathbb{R}^n), \, 1 < p < \infty$, by Coifman, Rochberg and Weiss


it was shown in


that, for all $0 < p < \infty$ and all $w \in A_\infty$,

$$\int_{\mathbb{R}^n} |[b; T]f(x)|^p w(x)dx \leq C \int_{\mathbb{R}^n} M^2 f(x)^p w(x)dx; \quad (11.1)$$

where $M^2 = M \circ M$. Hence, if $1 < p < \infty$ and $w \in A_p$, then $[b; T]$ is bounded on $L_p(w)$. 
Theorem

Let $b \in \text{BMO}$. Suppose that $w \in A_1$ such that

$$\sum_{k=1}^{\infty} \frac{\varphi(x, 2^k r) |B(x, 2^k r)|^\varepsilon}{w(B(x, 2^k r))} \leq C_1 \frac{\varphi(x, r) |B(x, r)|^\varepsilon}{w(B(x, r))}$$

for every $x \in \mathbb{R}^n$ and $r > 0$, and for some $\varepsilon > 0$. Then operator $[b; T]$ is bounded on $\mathcal{M}_{p, \varphi}(w)$. 
Theorem

Let $b \in \text{BMO}$. Suppose that $w \in A_1$ such that

$$\sum_{k=1}^{\infty} \varphi(x, 2^k r) \frac{|B(x, 2^k r)|^\varepsilon}{w(B(x, 2^k r))} \leq C_1 \frac{\varphi(x, r) |B(x, r)|^\varepsilon}{w(B(x, r))}$$

for every $x \in \mathbb{R}^n$ and $r > 0$, and for some $\varepsilon > 0$. Then operator $[b; T]$ is bounded on $\mathcal{M}_{p, \varphi}(w)$.

Corollary

Let $b \in \text{BMO}$, $0 < \kappa < 1$ and $w \in A_1$. Then $[b; T]$ is bounded on $\mathcal{M}_{p, \kappa}(w)$. 
The original inequality due to Coifman and Fefferman showed that Calderón-Zygmund singular integrals $T$ could be controlled by the Hardy-Littlewood maximal operator: more precisely, given $w \in A_\infty$,

$$\int_{\mathbb{R}^n} |Tf(x)|^p w(x) \, dx \leq C \int_{\mathbb{R}^n} Mf(x)^p w(x) \, dx.$$ 


Theorem

Let $1 < p < \infty$. Suppose that $w \in A_1$ such that

$$\sum_{k=1}^{\infty} \frac{\varphi(x, 2^k r)|B(x, 2^k r)|^\varepsilon}{w(B(x, 2^k r))} \leq C_1 \frac{\varphi(x, r)|B(x, r)|^\varepsilon}{w(B(x, r))}$$

for every $x \in \mathbb{R}^n$ and $r > 0$, and for some $\varepsilon > 0$. Then

$$\|Tf\|_{\mathcal{M}_p, \varphi(w)} \leq C\|Mf\|_{\mathcal{M}_p, \varphi(w)}.$$
Theorem

Let $1 < p < \infty$. Suppose that $w \in A_1$ such that

$$\sum_{k=1}^{\infty} \frac{\varphi(x, 2^k r) |B(x, 2^k r)|^\varepsilon}{w(B(x, 2^k r))} \leq C_1 \frac{\varphi(x, r) |B(x, r)|^\varepsilon}{w(B(x, r))}$$

for every $x \in \mathbb{R}^n$ and $r > 0$, and for some $\varepsilon > 0$. Then

$$\|Tf\|_{\mathcal{M}_p, \varphi(w)} \leq C \|Mf\|_{\mathcal{M}_p, \varphi(w)}.$$

Corollary

Let $1 < p < \infty$, $0 < \kappa < 1$ and $w \in A_1$. Then

$$\|Tf\|_{\mathcal{M}_p, \kappa(w)} \leq C \|Mf\|_{\mathcal{M}_p, \kappa(w)}.$$
Given a locally integrable function $f$, the Hardy-Littlewood maximal function, $Mf$, is defined by

$$Mf(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| \, dy,$$

where the supremum is taken over all balls $B$ in $\mathbb{R}^n$.

Given a locally integrable function $f$ the sharp maximal function $M^\# f$, is defined by

$$M^\# f(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| - \frac{1}{|B|} \int_B f |dy,$$

where the supremum is taken over all balls $B$ in $\mathbb{R}^n$.

We have the weighted variant of the Fefferman and Stein inequality, for all $p$, $0 < p < \infty$, and $w \in A_\infty$,

$$\int_{\mathbb{R}^n} Mf(x)^p w(x) \, dx \leq C \int_{\mathbb{R}^n} M^\# f(x)^p w(x) \, dx.$$

---

**Theorem**

Let $1 < p < \infty$. Suppose that $w \in A_1$ such that

$$\sum_{k=1}^{\infty} \frac{\varphi(x, 2^k r) |B(x, 2^k r)|^\varepsilon}{w(B(x, 2^k r))} \leq C_1 \frac{\varphi(x, r) |B(x, r)|^\varepsilon}{w(B(x, r))}$$

for every $x \in \mathbb{R}^n$ and $r > 0$, and for some $\varepsilon > 0$. Then

$$\|Mf\|_{\mathcal{M}_p, \varphi(w)} \leq C \|M^# f\|_{\mathcal{M}_p, \varphi(w)}.$$
Theorem

Let $1 < p < \infty$. Suppose that $w \in A_1$ such that

$$\sum_{k=1}^{\infty} \frac{\varphi(x, 2^k r) |B(x, 2^k r)|^\varepsilon}{w(B(x, 2^k r))} \leq C_1 \frac{\varphi(x, r) |B(x, r)|^\varepsilon}{w(B(x, r))}$$

for every $x \in \mathbb{R}^n$ and $r > 0$, and for some $\varepsilon > 0$. Then

$$\|Mf\|_{M_p, \varphi(w)} \leq C \|M^# f\|_{M_p, \varphi(w)}.$$

Corollary

Let $1 < p < \infty$, $0 < \kappa < 1$ and $w \in A_1$. Then

$$\|Mf\|_{M_p, \varphi(w)} \leq C \|M^# f\|_{M_p, \varphi(w)}.$$
Thank you for your attention.